32

THE JOURNAL

OF THE

INDIAN MATHEMATICAL SOCIETY

21050

NEW SERIES

Volume VI *No. 3.* *September 1942*

OFFICE OF THE REGISTRAR OF BOOKS
29 MAY 1943
MADRAS

CONTENTS

	Pages.
F. C. Auluck, S. Chowla, and H. Gupta: On the Maximum Value of the Number of Partitions of n into k parts ..	105—112
F. C. Auluck: An Asymptotic Formula for $p_k(n)$ ..	113—114
Hansraj Gupta: A Formula in Partitions ..	115—117
S. S. Pillai: On a Congruence Property of the Divisor Function ..	118—119
———: On the Divisors of a^n+1 ..	120—121
A. C. Choudhury: On Steiner's Polygons in a Web ..	122—126
C. N. Srinivasiengar: On the Quartic Developable ..	127—130
K. Rangaswami: The Geometry of Hart Tetrads ..	131—136
P. Kesava Menon: Multiplicative Functions which are Functions of the *g.c.d.* and *l.c.m.* of the Arguments ..	137—142
———: Transformations of Arithmetic Functions ..	143—152

MADRAS:
PRINTED AT THE MADRAS LAW JOURNAL PRESS, MYLAPORE.

The Indian Mathematical Society

Founded in 1907

President: R. VAIDYANATHASWAMI, M.A., D.SC. (University of Madras).

Treasurer: L. N. SUBRAMANIAM, M.A. (Madras Christian College, Madras).

Secretaries: S. MAHADEVA IYER, M.A., L.T. (Presidency College, Madras).

RAM BAHARI, M.A., PH.D. (St. Stephen's College, Delhi).

The Committee of Management includes the President, the Secretaries, the Treasurer and the following members:—

R. P. SHINTRE (Librarian), D. D. KOSAMBI, F. W. LEVI, A. NARASINGA RAO, A. N. SINGH, P. L. SRIVASTAVA M. R. SIDDIQUI and J. MACLEAN.

The Society publishes two quarterlies, the *Journal of the Indian Mathematical Society*, and the *Mathematics Student*, which are published in March, June, September and December. The *Journal* is managed by the following Board of Editors:—

R. VAIDYANATHASWAMI (*Chairman*)

K. ANANDA RAO
C. RACINE
D. D. KOSAMBI
N. R. SEN
F. W. LEVI
M. R. SIDDIQUI
S. S. PILLAI
T. VIJAYARAGHAVAN

C. N. SRINIVASIENGAR (*Secretary*)

1. Papers intended for the *Journal* should be addressed to C. N. Srinivasiengar, D.SC., College of Engineering, Bangalore, and papers intended for the *Student* should be addressed to A. Narasinga Rao, M.A., D.SC., Annamalai University, Annamalai Nagar.

2. Authors of papers appearing in the *Journal* will receive 50 reprints free of charge. Extra reprints can be had at cost price, on intimation.

3. The Secretary should be addressed to for back numbers of both the Quarterlies, as well as by intending members and subscribers.

4. Subscription for each of the Quarterlies is Rs. 6 per year. Single copy Rs. 1-8-0.

2105-0

ON THE MAXIMUM VALUE OF THE NUMBER OF PARTITIONS OF n INTO k PARTS.

BY

F. C. AULUCK, S. CHOWLA, and H. GUPTA.

[Received 26 January 1942.]

1. Let $p_k(n)$ denote the number of partitions of n into *exactly* k parts. We obviously have

$$\sum_{k=1}^{n} p_k(n) = p(n),$$

where $p(n)$ denotes the number of unrestricted partitions of n, a function introduced by Euler. Hardy and Ramanujan proved the asymptotic formula

$$p(n) \sim \frac{e^{c\sqrt{n}}}{(4\sqrt{3})n}, \text{ where } c = \pi\sqrt{\tfrac{2}{3}}.$$

They also proved a formula from which $p(n)$ can be estimated with great rapidity for fairly large values of n. D. H. Lehmer has used the Hardy-Ramanujan formula to calculate $p(14031)$. Recently Rademacher* has found an 'exact' formula for $p(n)$, suggested by the formula of Hardy and Ramanujan.

Some months ago, at the suggestion of Dr. Kothari, we undertook the study of $p_k(n)$. Although our first results were clumsy, our investigations got a fillip from the recent paper of Erdos and Lehner†, which contains the remarkable result that, denoting $\sum_{r=1}^{k} p_r(n)$ by $P_k(n)$,

$$\frac{P_k(n)}{p(n)} \sim \exp\left(-\frac{2}{c}e^{-\frac{1}{2}cx}\right),$$

* *Proceedings Nat. Acad. Sc.* 23 (1937), 78-84.

† *Duke Math. Jour.* 8 (1941), 335-45.

800-600-
129116

where $k = \frac{n^{\frac{1}{2}} \log n}{c} + xn^{\frac{1}{2}}$, and x is a fixed number (independent of n). This function $P_k(n)$ is a monotonic function of k, and is hence much easier to study than $p_k(n)$ of this paper. In fact the tables given at the end of this paper suggest that, regarding n as fixed and k as variable, $p_k(n)$ has a unique maximum in the sense that there exists a number k_0 such that

$$\left.\begin{aligned} p_k(n) &\geqslant p_{k-1}(n) \text{ for } k \leqslant k_0 \\ \text{and} \qquad p_k(n) &\leqslant p_{k-1}(n) \text{ for } k \geqslant k_0. \end{aligned}\right\} \qquad (A)$$

Generally there appears to be a unique k_0 but sometimes there is a consecutive set of numbers k_0 with the above property, e.g. $p_k(63)$ is maximum for $k = 13$, and $p_k(14)$ is maximum for $k = 4, 5$, the term *maximum* being used in the sense defined by (A). The value of k_0 suggested (but not proved) by the results of this paper, is asymptotic to $c^{-1}n^{\frac{1}{2}} \log n$. In fact the table shows that k_0 differs from $c^{-1}n^{\frac{1}{2}} \log n$ by a quantity which never exceeds one.

The results proved in this paper are:

THEOREM I

$$\frac{n^{\frac{1}{2}} p_k(n)}{p(n)} \sim \exp\left(-\tfrac{1}{2}cx - \frac{2}{c}e^{-\frac{1}{2}cx}\right),$$

where $$k = c^{-1}n^{\frac{1}{2}} \log n + xn^{\frac{1}{2}}.$$

THEOREM II

If k_1 is the value of k for which $p_k(n)$ is maximum (i.e., $p_k(n) \leqslant p_{k_1}(n)$ if $k \neq k_1$), then for $n > n_0$

$$n^{\frac{1}{2}} < k_1 < \delta n^{\frac{1}{2}} \log n,$$

where δ is any fixed number $> 1/c$.

It is clear that several problems remain open, e.g. the problem of the existence of k_0, and the problem of proving that $k_1 \sim c^{-1}n^{\frac{1}{2}} \log n$.

2. Proof of Theorem II.

We start with the identity

$$p_k(n) = p(n-k) - \sum_{1 \leqslant r \leqslant n-2k} p(n-k-\overline{k+r})$$

$$+ \sum_{\substack{0<r_1<r_2 \\ 1 \leqslant r_1+r_2 \leqslant n-3k}} p(n-k-\overline{k+r_1}-\overline{k+r_2}) - \ldots = p(n)\{S_1 - S_2 + S_3 - \ldots.\}.$$

This is, as in EL,* an application of the sieve of Eratosthenes. It is also a consequence of the formula

$$P_k(n) = \sum_{r=1}^{k} p_r(n) = p_k(n+k),$$

applied to the "sieve-formula" for $P_k(n)$ given in EL. This enables us to determine the values of $p_k(n)$ from a table of partitions. We give some simple cases:

$$p_k(n) = p(n-k) \text{ if } k \geqslant n/2, \qquad (B)$$

in all other cases

$$p_k(n) < p(n-k). \qquad (B')$$

For $\dfrac{n^{\frac{1}{2}} \log n}{c'} < k \leqslant n^{\frac{2}{3}}$ where $c' < c$,

$$\frac{p_k(n)}{p(n)} < \frac{p(n-k)}{p(n)} \sim \frac{n}{n-k} \exp\{c(n-k)^{\frac{1}{2}} - cn^{\frac{1}{2}}\}$$

$$\sim \exp\{-\tfrac{1}{2}ck/n^{\frac{1}{2}}\} < n^{-c/2c'},$$

therefore $$\frac{n^{\frac{1}{2}} p_k(n)}{p(n)} = o(1).$$

From (B') and the Hardy-Ramanujan formula we show that

$$\frac{n^{\frac{1}{2}} p_k(n)}{p(n)} = o(1) \text{ for } k \geqslant n^{\frac{2}{3}}.$$

This is also a consequence of Theorem I. It follows that k_1 of Theorem II satisfies $k_1 < \delta n^{\frac{1}{2}} \log n$, where δ is any constant $> 1/c$, i.e. the second half of Theorem II is proved.

*The paper of Erdos and Lehner will be referred to as EL.

To prove the first half, we use the following inequality,*

$$\frac{1}{k!}\binom{n-1}{k-1} < p_k(n) < \frac{1}{k!}\binom{n-1+k(k-1)/2}{k-1}. \qquad (C)$$

Then $\frac{1}{k!}\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!\,(n-k)!\,k!} < \frac{n^k}{(k!)^2}$, whose maximum is asymptotic to $e^{2\sqrt{n}}/Gn^{\frac{1}{2}}$, where G is a positive constant. For $k \leqslant n^{\frac{1}{2}}$

$$\binom{n-1+k(k-1)/2}{k-1} < \left\{1+\frac{k^2-k}{2(n-k+1)}\right\}^{k-1} < f(n),$$

where $f(n) \sim \exp\left\{ n^{\frac{1}{2}}(\log(\tfrac{3}{2})+\theta)\right\}$, where θ is any fixed positive constant. Hence when $k \leqslant n^{\frac{1}{2}}$, $p_k(n) < g(n)$, where

$$g(n) \sim n^{-\frac{1}{2}} \exp\left\{ n^{\frac{1}{2}} (\log \tfrac{3}{2}+2+\theta)\right\}.$$

But, by taking a suitable value of θ, $\log \frac{3}{2}+2+\theta < 2.5$, while $\pi\sqrt{\frac{2}{3}} > 2.5$. Thus

$$\frac{n^{\frac{1}{2}}\, p_k(n)}{p(n)} = o(1) \text{ for } k \leqslant n^{\frac{1}{2}},$$

which proves Theorem II.

3. Proof of theorem I. Consider values of k given by

$$k = c^{-1} n^{\frac{1}{2}} \log n + x n^{\frac{1}{2}}.$$

Evaluating $p_k(n)$ with the help of the sieve given in § 2, we get

$$S_1 = \frac{p(n-k)}{p(n)} \sim \frac{1}{n^{\frac{1}{2}}} \exp\left(-\tfrac{1}{2}cx\right).$$

$$S_2 \sim \sum \frac{n}{n-2k-r} \exp\left\{ c(n-2k-r)^{\frac{1}{2}} - cn^{\frac{1}{2}}\right\} = \sum_{\substack{1 \\ r\leqslant n^{\frac{3}{5}}}} + \sum_{\substack{2 \\ r>n^{\frac{3}{5}}}}.$$

In $\sum_1$, $\frac{n}{n-2k-r} \sim 1$ and $(n-2k-r)^{\frac{1}{2}} - n^{\frac{1}{2}} \sim -\frac{1}{2}(2k+r)/n^{\frac{1}{2}}$.

* See H. Gupta, *Proc. Ind. Ac. Sc.* 16 (1942), 101-2, and Auluck in this *Jour.* (current issue) pp. 113-4.

Thus
$$\sum\nolimits_1 \sim \frac{1}{n} e^{-cx} \sum_{1 \leqslant r \leqslant n^{\frac{3}{5}}} \exp\left(-\tfrac{1}{2}crn^{-\frac{1}{2}}\right)$$
$$= n^{-1}e^{-cx} \exp\left(-\tfrac{1}{2}cn^{-\frac{1}{2}}\right) \frac{1-\exp\left(-\frac{1}{2}cn^{\frac{1}{10}}\right)}{1-\exp\left(-\frac{1}{2}cn^{-\frac{1}{2}}\right)}$$
$$\sim 2c^{-1}n^{-1}e^{-cx}n^{\frac{1}{2}}.$$
$$\sum\nolimits_2 < n \sum_{r > n^{\frac{3}{5}}} \exp\left\{-\tfrac{1}{2}cn^{-\frac{1}{2}}(2k+r)\right\}$$
$$= n.n^{-1}e^{-cx} \sum_{r > n^{\frac{3}{5}}} \exp\left(-\tfrac{1}{2}crn^{-\frac{1}{2}}\right)$$
$$< e^{-cx} \exp\left(-\tfrac{1}{2}cn^{\frac{3}{5}}n^{-\frac{1}{2}}\right) \sum_{r > n^{\frac{3}{5}}} 1$$
$$< ne^{-cx} \exp\left(-\tfrac{1}{2}cn^{\frac{1}{10}}\right)$$
$$= o(1).$$

Therefore
$$S_2 \sim 2c^{-1}n^{-\frac{1}{2}}e^{-cx}.$$

Again
$$S_3 = \frac{1}{2!}\left[\frac{1}{p(n)} \sum_{1 \leqslant r_1 + r_2 \leqslant n-3k} p(n-3k-r_1-r_2)\right.$$
$$\left. - \frac{1}{p(n)} \sum_{1 \leqslant 2r \leqslant n-3k} p(n-3k-2r)\right] = \frac{1}{2!}\left[\sum\nolimits_1 + \sum\nolimits_2 - \sum_{r \leqslant n^{\frac{3}{5}}}\nolimits_3 - \sum_{r > n^{\frac{3}{5}}}\nolimits_4\right],$$

where Σ_1 runs over all pairs (r_1, r_2) in which neither r_1 nor r_2 exceeds $n^{\frac{3}{5}}$, Σ_2 over all pairs in which at least one number exceeds $n^{\frac{3}{5}}$.

$$\sum\nolimits_1 \sim n^{-\frac{3}{2}} \exp\left(-\frac{3c}{2}x\right) \sum_{r_1, r_2 \leqslant n^{\frac{3}{5}}} \exp\left[-\tfrac{1}{2}cn^{-\frac{1}{2}}(r_1+r_2)\right]$$
$$= n^{-\frac{3}{2}} \exp\left(-\frac{3c}{2}x\right) \left(\sum_{r_1 \leqslant n^{\frac{3}{5}}} \exp\left(-\tfrac{1}{2}cr_1n^{-\frac{1}{2}}\right)\right)^2$$
$$\sim n^{-\frac{3}{2}} \exp(-3cx/2)4n/c^2$$
$$= n^{-\frac{1}{2}}e^{-\frac{1}{2}cx}\left[2c^{-1}e^{-\frac{1}{2}cx}\right]^2.$$

$$\sum_2 < n^{-\frac{1}{2}} \exp(-3cx/2) \sum_{1\leqslant r_1 \leqslant n} \exp\left(-\tfrac{1}{2}cr_1 n^{-\frac{1}{2}}\right)$$

$$\times \sum_{r_2 > n^{\frac{3}{5}}} \exp\left(-\tfrac{1}{2}cr_2 n^{-\frac{1}{2}}\right)$$

$$< 2n^{-\frac{1}{2}} e^{-3cx/2}\, e^{-\frac{1}{2}cn^{-\frac{1}{2}}} c^{-1} n^{\frac{3}{2}} e^{-\frac{1}{2}cn^{1/10}}$$

$$= o(1).$$

$$\sum_3 \sim n^{-\frac{3}{2}} \exp(-3cx/2) \sum_{r \leqslant n^{\frac{3}{5}}} \exp\left(-crn^{-\frac{1}{2}}\right)$$

$$\sim n^{-\frac{3}{2}} \exp(-3cx/2) c^{-1} n^{\frac{1}{2}}$$

$$= o(1).$$

Similarly $\Sigma_4 = o(1)$.

Thus $S_3 \sim \dfrac{1}{2!}\,\dfrac{1}{n^{\frac{1}{2}}}\, e^{-\frac{1}{2}cx} \left[\dfrac{2}{c} \exp\left(-\tfrac{1}{2}cx\right)\right]^2.$

. . . .

$$S_{v+1} \sim \frac{1}{v!}\,\frac{1}{n^{\frac{1}{2}}}\, e^{-\frac{1}{2}cx} \left[\frac{2}{c} \exp\left(-\tfrac{1}{2}cx\right)\right]^v.$$

But $\qquad S_1 - S_2 + \ldots - S_{2v} < \dfrac{p_k(n)}{p(n)} < S_1 - S_2 + \ldots + S_{2v-1}$

and $S_v \to 0$ as $v \to \infty$. Hence

$$\frac{p_k(n)}{p(n)} \sim \sum_{v=1}^{\infty} (-)^{v-1} S_v \sim n^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}cx - 2c^{-1}e^{-\frac{1}{2}cx}\right],$$

which is Theorem I.

4. The above proof follows the method of EL, without however borrowing any results from that paper. A shorter proof is possible if we use the results of EL combined with the formula

$$P_k(n) = \sum_{r=1}^{k} p_r(n) = p_k(n+k).$$

Thus from EL

$$\frac{p_k(n+k)}{p(n)} \sim \exp\left(-2c^{-1}e^{-cx/2}\right),$$

where $k = c^{-1}n^{\frac{1}{2}}\log n + xn^{\frac{1}{2}}$. Hence changing n into $n-k$ (regarding k as fixed)

$$\frac{p_k(n)}{p(n-k)} \sim \exp\left(-\frac{2}{c}e^{-\frac{c}{2}x_1}\right),$$

where x_1 is defined by

$$k = \frac{(n-k)^{\frac{1}{2}}\log(n-k)}{c} + x_1(n-k)^{\frac{1}{2}}$$

$$= \frac{n^{\frac{1}{2}}\log n}{c} + O\left(\frac{\log n}{\sqrt{n}}\right) + x_1(n-k)^{\frac{1}{2}}.$$

We write the last expression as $c^{-1}n^{\frac{1}{2}}\log n + xn^{\frac{1}{2}}$,

so that $$x = x_1\left(\frac{n-k}{n}\right)^{\frac{1}{2}} + O\left(\frac{\log n}{n}\right).$$

But $k = O(n^{\frac{1}{2}}\log n)$ since x is fixed, and hence. $\lim_{n\to\infty} x/x_1 = 1$. Hence

$$\frac{p_k(n)}{p(n-k)} = \exp\left(-\frac{2}{c}e^{-\frac{1}{2}cx}\right),$$

where $$k = \frac{n^{\frac{1}{2}}\log n}{c} + xn^{\frac{1}{2}}.$$

It follows that

$$\frac{p_k(n)}{p(n)} \sim \frac{p(n-k)}{p(n)}\exp\left(-\frac{2}{c}e^{-cx/2}\right)$$

$$\sim n^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}cx - 2c^{-1}e^{-\frac{1}{2}cx}\right),$$

where k is defined above.

5. The following table gives the values of k for which $p_k(n)$ is maximum, and the corresponding values of $p_k(n)$.

n	k	$p_k(n)$	n	k	$p_k(n)$
1	1	1	33	8	1090
2	1	1	34	8	1297
3	1	1	35	9	1549
4	2	2	36	9	1845
5	2	2	37	9	2194
6	2,3	3	38	9	2592
7	3	4	39	9	3060
8	3,4	5	40	10	3590
9	3	7	41	10	4242
10	4	9	42	10	5013
11	4	11	43	10	5888
12	4	15	44	10	6912
13	4,5	18	45	10	8070
14	4,5	23	46	10	9418
15	5	30	47	11	11004
16	5	37	48	11	12866
17	5	47	49	11	15021
18	6	58	50	11	17475
19	6	71	51	11	20298
20	6	90	52	11	23501
21	6	110	53	11	27169
22	6	136	54	12	31570
23	7	164	55	12	36578
24	7	201	56	12	42333
25	7	248	57	12	48849
26	7	300	58	12	56297
27	7	364	59	12	64707
28	7	436	60	13	74331
29	8	525	61	13	85711
30	8	638	62	13	98609
31	8	764	63	13	113287
32	8	919			

AN ASYMPTOTIC FORMULA FOR $p_k(n)$

BY

F. C. AULUCK, *Delhi University*

[Received 26 January 1942]

1. The object of this note is to provide an elementary proof for the following theorem which was first proved by Erdos and Lehner,* and recently by Hansraj Gupta† by an independent method.

THEOREM.

$$p_k(n) \sim \frac{1}{k!} \binom{n-1}{k-1}$$

for all values of k such that $k = o(n^{\frac{1}{3}})$.

2. Consider‡ *n unit* digits as constituting the number n, and let there be k compartments. The number of ways, hereafter referred to as *complexions*, in which the n *units* can be distributed among the k compartments, each compartment containing at least one *unit*, is equal to the number of ways of collecting $n-k$ objects into k compartments, any particular compartment being used any number of times (from 0 to $n-k$); and is therefore equal to

$$_kH_{n-k} = \binom{n-1}{k-1}. \qquad (1)$$

Since any one partition of n into k *different* parts will generate $k!$ complexions, and a partition in which all the summands are not different will generate less than $k!$ complexions, we have

**Duke Math. Jour.* 8 (1941), 335-45.

†*Pro. Ind. Ac. Sc.* 16 (1942), 101-2.

‡This method was suggested to me by Dr. D. S. Kothari.

$$p_k(n)\,.\,k! > \binom{n-1}{k-1}. \tag{2}$$

3. However, any partition belonging to $p_k(n)$ can be transformed into a partition (of a number larger than n) of k *different* summands by adding a suitable number of *units* to the various summands of the original partition.* The partition of n in which all summands are equal will require $c_k = k(k-1)/2$ *units* to make all its summands unequal, and any other member of $p_k(n)$ will require a smaller number of *units* than c_k. But, by adding c_k *units*, every partition of n could be transformed into a partition of $n+c_k$, with all its summands different—the number of *units* left over after adding the minimum number required to transform any partition of n is to be added to the largest summand in the partition. Thus we have

$$k!\, p_k(n) < {}_kH_{n-k+c_k}. \tag{3}$$

From (2) and (3) we get Gupta's inequality

$$\binom{n-1}{k-1} < k!\, p_k(n) < \binom{n+c_k-1}{k-1}.$$

Since $\binom{n+c_k-1}{k-1} < \binom{n-1}{k-1} \exp(k^3/2n)$ for $k^2 < n$, we have

$$1 < \frac{k!}{\binom{n-1}{k-1}}\, p_k(n) < \exp(k^3/2n), \tag{4}$$

provided $k < n^{\frac{1}{2}}$. Hence

$$p_k(n) \sim \frac{1}{k!}\binom{n-1}{k-1} \text{ for } k = o(n^{\frac{1}{3}}). \tag{5}$$

* This is easily seen by arranging the k summands of the partition in an *ascending* order and then successively adding 0, 1, 2, ..., $(k-1)$ to them.

A FORMULA IN PARTITIONS

BY

HANSRAJ GUPTA, *Hoshiarpur*

[Received 27 January, 1942]

1. In what follows, $P_m(n)$ denotes the number of those partitions of n in which the smallest element is m; $p_m(n)$ the number of those in which m is the largest element, i.e. the number of partitions of n into exactly m non-zero parts; and $p(n)$ denotes the number of unrestricted partitions of n.

It is well known that

$$p_m(n) = p_1(n-m) + p_2(n-m) + p_3(n-m) + \dots + p_m(n-m), \quad (1)$$

$$= p_{m-1}(n-1) + p_m(n-m); \quad (2)$$

and $\quad p_1(n) = 1,\ p_2(n) = [n/2],\ p_3(n) = [(n^2+3)/12]; \quad (3)$

where $[x]$ denotes as usual the greatest integer in x.

Making use of (2), I have recently shown* that

$$\frac{1}{m!}\binom{n-1}{m-1} \leqslant p_m(n) \leqslant \frac{1}{m!}\binom{n+\frac{1}{2}m(m-1)}{m-1}. \quad (4)$$

In § 2, I establish a relation between the two types of restricted partitions mentioned above, and deduce from it certain results proved by me in an earlier paper†. In § 3, I prove a result similar to (4) for $P_m(n)$. This leads to an asymptotic formula for $P_m(jm+m)$, which is very similar to the one given by Erdos and Lehner‡ for $p_m(n)$.

2. THEOREM 1. *For values of* $n \geqslant 1$, $m \geqslant 0$;

$$P_{m+1}(n+2m+1) = p_1(n) + p_2(n-m) + p_3(n-2m) + \dots$$
$$+ p_{r+1}(n-rm), \textit{ where } r = [(n-1)/(m+1)]. \quad (5)$$

*Proc. Ind. Ac. Sc. 16 (1942), 101-2.

†'On Partitions of n', *Jour. Lond. Math. Soc.* 11 (1936), 278-80.

‡*Duke Math. Jour.* 8 (1941), 335-45.

PROOF. Suppose $(n+2m+1)$ is partitioned into $(j+1)$ parts exactly, $j \geqslant 1$, one of the parts being $(m+1)$ and each of the others being either greater than or equal to $(m+1)$. If we diminish each of the $(j+1)$ parts by $(m+1)$, we are left with a partition of t, where

$$t=(n+2m+1)-(m+1)(j+1),$$

into at most j parts. The number of partitions therefore, is

$$p_1(t)+p_2(t)+p_3(t)+\ldots+p_j(t)=p_j(t+j)=p_j(n+m-jm).$$

Giving to j the values 1, 2, 3,..., we obtain the result stated above.

Since $\quad n+m-jm \geqslant j,\ j \leqslant (n+m)/(m+1).$

As particular cases of (5), we have for $0 \leqslant s \leqslant m$,

$$P_{m+1}(2m+2+s)=p_1(s+1)=1;$$

$$\begin{aligned} P_{m+1}(3m+3+s) &= p_1(m+s+2)+p_2(s+2), \\ &= 1+[\tfrac{1}{2}(s+2)]=2+[\tfrac{1}{2}s]; \end{aligned}$$

$$\begin{aligned} P_{m+1}(4m+4+s) &= p_1(2m+s+3)+p_2(m+s+3)+p_3(s+3), \\ &= 1+[\tfrac{1}{2}(m+s+3)]+[\tfrac{1}{12}(\overline{s+3}\ +3)] \\ &= 3+[\tfrac{1}{2}(m+s+1)]+[\tfrac{1}{12}(s^2+6s)] \end{aligned}$$

and $P_1(n+1)=p_1(n)+p_2(n)+p_3(n)+\ldots+p_n(n)=p(n)$.

3. We proceed to prove

THEOREM 2. *For values of* $j \geqslant 2$, $m \geqslant 1$, *and* $0 \leqslant s \leqslant (m-1)$,

$$\frac{1}{j!}\binom{s+j-1}{j-1}+\frac{1}{(j-1)!}\binom{m+s+j-2}{j-2} \leqslant P_m(jm+m+s)$$

$$\leqslant \frac{1}{j!}\binom{u}{j-1}+\frac{1}{(j-1)!}\binom{m+u+c_j}{j-2}, \qquad (6)$$

where u *stands for* $s+\frac{1}{2}j(j+1)$, *and* c_j *is a certain function of* j *alone.*

We have

$$P_m(jm+m+s)=\sum_{k=0}^{j-1} p_{j-k}(km+s+j-k):$$

This is
$$\geqslant p_j(s+j)+p_{j-1}(m+s+j-1),$$
$$\geqslant \frac{1}{j!}\binom{s+j-1}{j-1}+\frac{1}{(j-1)!}\binom{m+s+j-2}{j-2};$$
and is
$$\leqslant \frac{1}{j!}\binom{u}{j-1}+\sum_{k=1}^{j-1}\frac{1}{(j-k)!}\binom{km+s+\frac{1}{2}(j-k)(j-k+1)}{j-k-1}. \quad (7)$$

The Σ in (7) is
$$< \sum_{k=1}^{j-1}\frac{1}{(j-k)!}\binom{km+u-j}{j-k-1}$$
$$< \sum_{k=1}^{j-1}\frac{1}{(j-k)!\,(j-k-1)!}(km+u-j)^{j-k-1};$$
$$< \sum_{k=1}^{j-1}\frac{k^{j-k-1}}{(j-k)!\,(j-k-1)!}(m+u-j)^{j-k-1};$$
$$< \frac{1}{(j-1)!\,(j-2)!}\{(m+u-j)^{j-2}+(j-1)(j-2)\,2^{j-3}$$
$$\times(m+u-j)^{j-3}+(j-1)(j-2)^2(j-3)\,3^{j-4}(m+u-j)^{j-4}+\dots$$
$$+(j-1)!\,(j-2)!\},$$
$$< \frac{1}{(j-1)!\,(j-2)!}\{(m+u-j)+(j-1)\,2^{j-3}\}^{j-2}$$
$$< \frac{1}{(j-1)!}\binom{m+u+(j-1)\,2^{j-3}}{j-2}.$$

Hence $c_j=(j-1)\,2^{j-3}$ satisfies the right hand inequality in (6).

Since
$$\binom{m+s+j-2}{j-2}\sim\binom{m+u+c_j}{j-2}$$
if
$$(j-2)\{c_j+\tfrac{1}{2}j(j+1)\}=o(m),$$
i.e. if
$$j=o(\log m)$$
we have
$$P_m(jm+m+s)\sim\frac{1}{j!}\binom{u}{j-1}+\frac{1}{(j-1)!}\binom{m+u+c_j}{j-2}; \quad (8)$$
and in particular
$$P_m(jm+m)\sim\frac{1}{(j-1)!}\binom{m+c_j}{j-2}; \quad (9)$$
in the range $j=o(\log m)$.

ON A CONGRUENCE PROPERTY OF THE DIVISOR FUNCTION

BY

S. S. PILLAI, *Calcutta.*

[Received 2 December 1941]

Let $d(n)$ denote the number of divisors of n, and $N(k, x)$ the number of $n \leqslant x$, for which $d(n)$ is a multiple of k. The object of this paper is to obtain an asymptotic formula for $N(k, x)$ when k is a prime.

The following notations are used:
k is a prime; $f(n)$ = the number of different prime factors of n; $t = t(n) = f(n) - 1$; $\phi(n)$ = Euler's ϕ function; q denotes a square free number; $\nu = 1/(k-1)$; r runs through 1, 2, 3, . . . $F(m)$ denotes the number of $n \leqslant x$, which is a multiple of m and is such that $(m, n/m) = 1$; Q runs through those square-free numbers whose prime factors do not exceed x^{ν}.

LEMMA 1. *If $G(m, x)$ denotes the number of numbers $\leqslant x$ and prime to m, then* $G(m, x) = \dfrac{x\phi(m)}{m} + O(2^{f(m)})$.

Let $p_1, \ldots, p_r$ denote the different prime factors of m. Then

$$G(m, x) = [x] - \sum \left[\frac{x}{p_1}\right] + \sum \left[\frac{x}{p_1 p_2}\right] - \ldots,$$

where $[x]$ denotes the integral part of x.

So

$$G(m, x) = x - \sum \frac{x}{p_1} + \sum \frac{x}{p_1 p_2} - \ldots + O(2^r)$$
$$= \frac{x\phi(m)}{m} + O(2^{f(m)}).$$

Now we are in a position to prove our result. If $(a, p) = 1$, where p is a prime, then $d(ap^{rk-1}) \equiv 0 \pmod{k}$. Hence

$$\begin{aligned} N_k(x) &\geqslant \sum_r F(2^r), \\ &\leqslant \sum F(2^r) + \sum F(3^r), \\ &\geqslant \sum F(2^r) + \sum F(3^r) - \sum F(2^r, 3^r) \end{aligned}$$

and so on. Hence

$$N(k, x) = \sum_{q, r} (-1)^{t(q)} F(q^{rk-1}).$$

From LEMMA (1), $F(q^{rk-1}) = x\phi(q)/q^{rk} + O(2^{f(q)})$,
and directly $F(q^{rk-1}) = O(x/q^{rk-1})$.

So $N(k, x)$

$$\begin{aligned} &= \sum_{q^r \leqslant x^\nu} (-1)^{t(q)} F(q^{rk-1}) + \sum_{q^r > x^\nu} (-1)^{t(q)} F(q^{rk-1}) \\ &= x \sum_{q^r \leqslant x^\nu} \frac{(-1)^{t(q)}\phi(q)}{q^{rk}} + O \sum_{q^r \leqslant x^\nu} 2^{f(q)} + O(x) \sum_{q^r > x^\nu} \frac{1}{q^{rk-1}} \\ &= x \sum_{q, r} \frac{(-1)^{t(q)}\phi(q)}{q^{rk}} + O(x) \sum_{n > x^\nu} \frac{1}{n^{k-1}} + O \sum_{n \leqslant x^\nu} d(n) + O(x) \sum_{n > x^\nu} \frac{1}{n^{k-1}} \\ &= x \, . \, A + O(x^\nu) + O(x^\nu \log x) + O(x^\nu) \\ &= x \, . \, A + O(x^\nu \log x). \end{aligned}$$

If $k \geqslant 3$, $A = \sum_{q, r} \frac{(-1)^{t(q)}\phi(q)}{q^{rk}}$ is convergent. Hence we get

THEOREM I. *When k is prime and $\geqslant 3$, $N(k, x) = A \, . \, x + O(x^{\frac{1}{k-1}} \log x)$.*

If $d(n)$ is odd, n should be a perfect square. So we get

THEOREM II. $N(2, x) = [x] - [\sqrt{x}]$.

ON THE DIVISORS OF a^n+1

BY

S. S. PILLAI, *Calcutta*

[Received 2 December, 1941]

Are there infinitely many primes which do not divide a^n+1 for any n? The object of this note is to answer this question.

In this note, p always stands for an odd prime, and in all the congruences the modulus is p. So the modulus will not be mentioned. The answer depends upon the following two lemmas which are almost trivial, but at the same time interesting.

LEMMA 1. *If $a^m\equiv 1$, where m is odd, then the congruence $a^x\equiv -1$ has no solution.*

If possible, let $a^n\equiv -1$. (1)

Then $a^{2n}\equiv 1$. (2)

But $a^m\equiv 1$. (3)

From (2) and (3), we get that, if $d=(2n, m)$, then

$$a^d\equiv 1. \tag{4}$$

Since m is odd, d is also odd. So d divides n.

Let $n=ld$. Then from (4),

$$a^n\equiv +1 \tag{5}$$

From (1) and (5), $2\equiv 0$.

Since p is an odd prime, (6) is impossible. (6)

So a^n+1 is not divisible by p for any n.

This lemma means that no prime factor $a^{2r+1}-1$ divides a^n+1 for any n; and the question is answered in a sense.

LEMMA 2 *If $p=4ma-1$, then a is a quadratic residue of p.*

Let $a = 2^r b$, where b is odd.

(i) Let $b = 4s-1$.

Then $(b\,|\,p)(p\,|\,b) = -1$.

So $(b\,|\,p) = -1/(p\,|\,b) = -1/(-1\,|\,b) = -1/-1 = +1$.

(ii) Let $b = 4s+1$.

Then $(b\,|\,p)(p\,|\,b) = +1$.

So $(b\,|\,p) = 1/(p\,|\,b) = 1/(-1\,|\,b) = 1/1 = +1$.

(ii) When a is even, $p = 8s-1$. So $(2^r\,|\,p) = (2\,|\,p) = +1$.

Therefore,

$(a\,|\,p) = (2^r b\,|\,p) = (2^r\,|\,p)(b\,|\,p) = (+1)(+1) = +1$.

So the lemma is proved.

Cor. 1. *If $4ma-1$ is prime, $a^{2ma-1}+1$ is not divisible by $4ma-1$.*

Cor. 2. *If $4m-1$ is prime, then all the divisors of m are quadratic residues of $4m-1$.*

Theorem. *No prime of the form $4am-1$ divides a^n+1 for any n.*

Let $p = 4am-1$. Then, by Lemma 2, a is a quadratic residue of p. So $a^{2am-1} \equiv 1$, for $2am-1 = \frac{1}{2}(p-1)$.

Therefore, by Lemma 1, the congruence $a^x \equiv -1$ has no solution. Hence the theorem is proved. A particular case is interesting:

No prime of the form $8m-1$ divides 2^n+1 for any n.

If $p = 8m \pm 3$, 2 is a non-quadratic residue of p; so p divides 2^n+1, for suitable n.

But I am not able to prove that there are infinitely many primes of the form $8m+1$, which do not divide 2^n+1 for any n. 73 and 89 are two such primes.

ON STEINER'S POLYGONS IN A WEB

BY

A. C. CHOUDHURY, *Calcutta*

[Received 20 March, 1942]

1. The geometry of a 3-web is closely related to the geometry of curves of the third order and of the third class. Indeed, the theorem of Graf and Sauer states that the hexagonal 3-webs of straight lines are exactly the webs formed by the straight lines of a curve of the third class. So one expects that many other properties of a curve of the third class may be interpreted as properties of a web. In this paper it has been shown that the notion of a Steiner polygon in a cubic curve may be considered as a geometrical expression of the equation $a^n = 1$ in a 3-web in which the multiplication is given by Thomsen's scheme.

A web is, after G. Thomsen, a set of two types of elements called "points" and "lines" in which lines are divided into three classes, each class being called a pencil. For the elements of a web the following axioms hold:

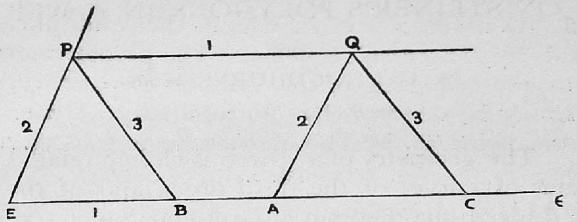
1. Through each point there passes exactly one line of each pencil.

2. Two lines of the same pencil have no point and two lines of different pencils have exactly one point, in common.

He has defined multiplication in a web in the following way:—Choose from the web a fixed line ε, say of the first pencil and a fixed point E on it. Then each point of our line corresponds to a symbol in such a manner that the point A corresponds to the symbol a, the point B to the symbol b, and the point E to the symbol e.

Then with the help of our web, a multiplication of these symbols is defined (as given schematically in fig. 1),

Fig. 1.



so that the point P is obtained at first by the intersection of the 2-line through E and the 3-line through B. Then draw through P the 1-line which is intersected by the 2-line through A at Q. Draw the 3-line through Q which intersects ε in a point C. If c is the symbol corresponding to the point C, then $c = a . b$. Evidently by a similar construction we can find the point $c' = b . a$, and this point is different from c. On the other hand if c and one of the elements a, b are given, the other element can be found out from the same figure. Thus the left inverse and the right inverse elements are defined.

Suppose that A is a multiplicative system of elements in which

(i) there exists a unit element e such that

$$ae = e . a = a$$

(ii) the equations $a . x = b, y . a = b$

are always solvable for arbitrary choice of a and b.

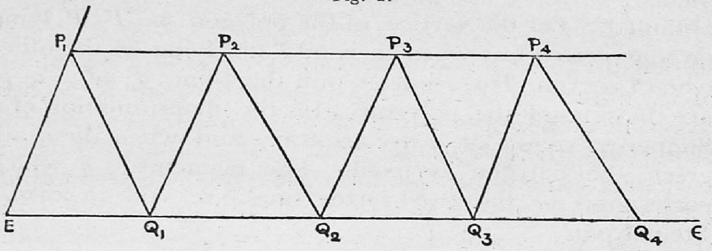
Such a multiplicative system will be called a *quasi-group*. To a web there belongs a quasigroup and conversely. The relation between the two is the following:

To each point of the web, there corresponds a pair of elements (a_1, a_2). All points for which a_1 has the same value form a 2-line and all points for which $a_2 . a_1$ is fixed lie on a 3-line.

The quasigroup belonging to a web is uniquely fixed if the point E is fixed. If the point E is altered we get a quasigroup which is in general not isomorphic to the former.

2. Consider a web in which the multiplication is defined by Thomsen's scheme. The points corresponding to the sequence of the symbols $a^n = a^{n-1}.a$, $a^1 = a$, $a^0 = e$ may be denoted by Q_n and may be constructed from the point Q_1 by Thomsen's scheme (see fig. 2).

Fig. 2.



The sequence of symbols a^n may or may not be all different. If they are not all different, then $a^m = a^n$. The corresponding points Q_m and Q_n are identical. We say that the points Q_i define a Steiner polygon. Q_i are called the vertices of the polygon, and the fundamental 1-line ε will be called the vertex line. The points P_i and the 1-line on which they lie will be called cognate vertices and vertex line. As $Q_n = Q_m$, P_{n+1} will be identical with P_{m+1} and hence both the vertex lines will be closed.

An example of Steiner's polygon is obtained in the web formed by the pencils

$$r = m; \ \theta = 2n\pi/p, \ r + p\theta/2\pi = m + n,$$
$$(1 \leqslant m \leqslant p, \ 1 \leqslant n \leqslant p),$$

m, n being integers. Let $r = p$ be the fundamental 1-line ε and the point E be its intersection with the line $\theta = 2\pi$. Each point of the web is given by the symbol (m, n). If $(p, 1) = a$, then $(p, 2) = a^2$, $(p, 3) = a^3, \ldots, (p, p) = a^p$.

In the above figure, the vertex line of the Steiner polygon is of a special nature. A Steiner polygon in which the vertex lines are arbitrarily given 1-lines may be constructed by the following sequence of points Q_i. Let the symbol corresponding to Q_0 be c_0 and p, q be two arbitrary elements. Then the symbols corresponding to the points Q_i, $i > 0$ will be c_i where $c_i . p = c_{i+1} q$.

The sequence c_i may or may not contain a finite number of elements. If it contains a finite number of elements, then $c_m = c_n$ and a closed Steiner polygon is obtained. Let the vertices of the polygon be P_i, P_i lying on one vertex line when i is odd and lying on the other when i is even. The 2-lines c_i and the 3-lines d_i, $(d_i = c_i . p)$ are the sides of the polygon. In the determination of a Steiner po'ygon c_0, p, q are arbitrary and when these are given, the polygon is fixed. The elements p, q which correspond to the two vertex lines are said to form a Steiner pair.

3. In a non-associative system, the powers a^n of an element a do not exist in the usual sense. They are defined by means of the sequence $a^n = a^{n-1} . a$. Therefore a Steiner polygon in a web is associated with the idea of the powers a^n in the corresponding non-associated system. If every Reidemeister's figure is closed in the web, then the corresponding system of the symbols forms a group; on the other hand if every Thomsen's triangle is closed, then the system of the corresponding symbols forms an abelian group. In such a web, many propositions about a Steiner polygon in a cubic curve become self-evident and may at once be stated in the language of the web.

For simplicity let c_0 be taken to be e where e is the identity element of the group. Then the sequence giving the polygon is

$$c_i . p = c_{i+1} q, \quad c_0 = e$$

i.e. $c_{i+1} = c_i \, pq^{-1}$ and therefore $c_n = p^n q^{-n}$.

The condition that the pair p, q may enclose a polygon of $2n$ sides is then $p^n q^{-n} = e$, i.e. $p^n = q^n$. Therefore

$$(ap)^n = (aq)^n.$$

Thus if (p, q) is a Steiner pair, then (ap, aq) is also a Steiner pair of the same order. As a property of a cubic curve, this has been stated in the form that if a Steiner pair is projected from any point of the cubic, then the projections form another Steiner pair of the same order. Again if $p^{2n} = q^{2n}$, then $p^n = q^n$. This gives us that if the tangentials of two given points of a cubic form a Steiner pair of order n, then the points themselves form a Steiner pair of the same order. The statement becomes evident in a web if we define the 1-line p^2 as the tangential of the 1 line p. The 3-line p^2 passes through the point (p, p) of the web, and this shows that the definition given here conforms to the definition of the tangential of a point on the cubic curve. Lastly if $p^n = q^n$, then $(p^{\frac{1}{2}} q^{\frac{1}{2}})^{2n} = p^{2n} = q^{2n}$, provided $p^{\frac{1}{2}}, q^{\frac{1}{2}}$ exist. Thus if P, Q are a Steiner pair of order n, R the point where the line PQ meets the cubic, and S the point of contact of one of the tangents from R, then P, S and Q, S are Steiner pairs of order $2n$.

REFERENCES

1. W. Blaschke und G. Bol:—*Geometrie der Gewebe*, 32-42.
2. Hilton: *Theory of plane curves*, Chap. XII.

OFFICE OF THE REGISTRAR
29 MAY 1943
MADRAS

600 008
129116

ON THE QUARTIC DEVELOPABLE*

BY

C. N. SRINIVASIENGAR, *Bangalore*

1. The quartic developable is generated by the tangents of a twisted cubic. Taking the cubic as

$$x = t^3,\ y = t^2,\ z = t,\ w = 1, \qquad (1)$$

the equation of the developable is

$$f(x, y, z, w) \equiv 6xyzw - 4y^3w - 4xz^3 + 3y^2z^2 - x^2w^2 = 0. \qquad (2)$$

The object of this paper is to obtain a remarkable property regarding the first polar surfaces of this developable.

The first polar surface of the point $(t^3, t^2, t, 1)$ with respect to (2) is given by

$$F \equiv t^3(3yzw - 2z^3 - xw^2) + 3t^2(xzw - 2y^2w + yz^2)$$
$$+ 3t(xyw - 2xz^2 + y^2z) + (3xyz - 2y^3 - x^2w) = 0. \qquad (3)$$

For the particular values $t = 0$ and $t = \infty$, the surface $F = 0$ is at once recognized to be a Cayley cubic scroll. Since the equations of a twisted cubic can be expressed in the form (1), so as to have the parameters of any two points on it as 0 and ∞, it follows that the above property holds good for every point on the twisted cubic. It is however worth while to reduce the equation (3) to the standard equation of a Cayley cubic. By working out the values of $\partial F/\partial x$, $\partial F/\partial y$, etc. we verify that (3) possesses a double line given by

$$x - 2ty + t^2z = 0, \quad y - 2tz + t^2w = 0.$$

It is verified that this line is the tangent to the twisted cubic at the point 't'. Put

$$x' = x - 2ty + t^2z;\ y' = y - 2tz + t^2w.$$

*Read at the conference of the Indian Mathematical Society held at Aligarh in December 1941.

JL
Bm21,N34

Reducing F by these substitutions, we obtain

$$F \equiv 3y'(x'-ty')(z-tw)-2y'^3-w(x'-ty')^2.$$

Hence the transformation

$$X \equiv x'-ty' = x-3ty+3t^2z-t^3w, \quad Y \equiv y' = y-2tz+t^2w,$$
$$Z \equiv z-tw, \quad W \equiv w \qquad (4)$$

reduces $F=0$ to the form

$$3XYZ-2Y^3-WX^2=0. \qquad (5)$$

The generators of the surface (5) are given by

$$Y=mX, \; 2m^3X-3mZ+W=0, \qquad (6)$$

where m is arbitrary. The generators belong to the two linear complexes $P_{12}=0$, and $3P_{23}=P_{14}$, where the capital P's are line coordinates referred to the coordinate system X, Y, Z, W. From (4), we obtain

$$P_{12}=p_{12}+3t^2p_{23}+t^2p_{14}-2t^3p_{24}-2tp_{13}+t^4p_{34}$$
$$P_{23}=p_{23}+t^2p_{34}-tp_{24}, \; P_{14}=p_{14}-3tp_{24}+3t^2p_{34},$$

where the p's are line coordinates referred to x, y, z, w. Hence the equation $3P_{23}=P_{14}$ reduces to $3p_{23}=p_{14}$, and is thus independent of t. We have thus the following

THEOREM 1. *The first polar surfaces of points on a twisted cubic with respect to its tangent developable form a family of Cayley cubic scrolls. The generators of all these scrolls belong to the same linear complex as the generators of the quartic developable.*

2. We next proceed to examine where any particular generator given by (6) meets the quartic developable. The inverse of the transformation (4) is

$$x=X+3tY+3t^2Z+t^3W, \quad y=Y+2tZ+t^2W, \quad z=Z+tW,$$
$$w=W. \qquad (7)$$

Substituting these in (2), we obtain on simplification

$$6XYZW-4XZ^3-X^2W^2+3Y^2Z^2-4Y^3W=0. \qquad (8)$$

In other words, *the expression* $6xyzw-4xz^3-x^2w^2+3y^2z^2-4y^3w$ *is an invariant for the transformation* (7) *for all values of t.* (7) in fact defines a continuous one-parameter

group of transformations for which the infinitesimal transformation is given by $X\phi \equiv 3y\partial\phi/\partial x + 2z\partial\phi/\partial y + w\partial\phi/\partial z$, with the usual symbolic notation. It is at once verified that $Xf = 0$ when f is given by (2).

We now solve (6) and (8), and we obtain $X = 0$, $m^3X = W$ thrice. $X = 0$ gives a point on the tangent at 't' to the twisted cubic, this tangent being in fact the double line of the Cayley cubic in question. The other solution $m^3X = W$ gives the point $(1/m^3, 1/m^2, 1/m, 1)$ in the X, Y, Z, W coordinates-system, i.e. $(t+1/m)^3$, $(t+1/m)^2$, $(t+1/m)$, 1, in the x, y, z, w system. We have thus

THEOREM 2. *Every generator of any of the Cayley cubics defined in Theorem 1 meets the quartic developable in three coincident points at a point on the twisted cubic, and therefore lies in the osculating plane of the curve at that point.*

This result also follows easily by using the property* that *the edge of regression of any developable is an asympotic curve on the first polar surface of any point in space.* For, if P be any point on the twisted cubic, the generator through P of any of the Cayley cubics, being a tangent to this surface, must lie in the osculating plane of the curve, since the curve is an asymptotic line on the Cayley cubic. But this plane is also the unodal plane to the developable, and hence the generator meets the developable thrice at P. The remaining point of intersection is on the tangent at 't' to the twisted cubic.

3. We next consider the first polar surface of the point (0, 1, 0, 0), whose equation is $xzw + yz^2 - 2y^2w = 0$. This surface is a non-ruled cubic having a conic node at (0, 0, 0, 1) and a binode at (1, 0, 0, 0) such that one of the binodal planes touches the surface at all points

*See my paper, 'Some properties of developable surfaces,' *Jour. Indian Math. Soc.* (1) 18 (1930), 199-200. The discussion has been confined there to Cartesian coordinates, but the results can be easily worked out for general homogeneous coordinates.

along the binodal edge $z = 0$, $w = 0$. This surface has been styled as the cubic surface (B_5, C_2).*

Now the point (0, 1, 0, 0) is the point where the tangent to the twisted cubic at A (1, 0, 0, 0) meets the osculating plane at D (0, 0, 0, 1). Conversely, starting from any point Q on the developable, we can choose the point A as the point where the generator through Q meets the edge of regression. Out of the three osculating planes of the twisted cubic that pass through Q, two coincide with the osculating plane at A. Let the foot of the other plane be taken as D. The representation of the twisted cubic in the form (1) can be effected so as to have A and D chosen in this way as the vertices (1, 0, 0, 0) and (0, 0, 0, 1) of the tetrahedron of reference. Hence the property obtained above with regard to the polar surface of (0, 1, 0, 0) is characteristic of any point on the quartic developable, not lying on the edge of regression. We have thus

THEOREM 3. *The first polar surface with respect to a quartic developable, of any point on the surface not lying on the edge of regression is a cubic surface of the type (B_5, C_2) which possesses a binode at the foot of the generator through the point, and a conic node at the foot of the second osculating plane through the point. The generator of the developable through the point lies on the first polar, and one of the biplanes of the first polar touches it all along this line.*

4. The polar quadric of any point on the deve'opable is a cone. The Hessian of any developable of order n consists of the developable itself, together with a surface of order $3n-8$ which has been called the pro-Hessian. We may conclude by observing that *for the quartic developable, the pro-Hessian coincides with the Hessian.* It is verified by actual evaluation that the Hessian of the surface $f = 0$ given by (2) is $H = 432 f^2$.

*See Cayley, *Collecte l papers,* Vol. 6, p. 437 and Henderson, *Twentysev:n Lines on a cubic surface,* p. 79.

THE GEOMETRY OF HART TETRADS*

BY

K. RANGASWAMI,† *Annamalainagar*

[Received 2 April, 1942]

1. It is well known that given three circles a, b, c in a plane we can associate with them, in eight ways, another circle d such that the four circles a, b, c, d are all touched by four other circles, in which neither system of circles admits of a common orthogonal circle. Two such systems of circles are called *Complementary Hart Systems*. While it is plain that certain geometrical relations should exist between the four circles of a Hart tetrad, it should be admitted that the relations between the circles considered as a plane configuration are of a fairly complicated nature and are best studied by means of the space representation of circles, in which the point circles of the plane correspond to points on a quadric surface, called the *Absolute*. Thus, if we follow the usual practice and use the language of three-dimensional geometry so that our symbols represent both points in the representative space and the corresponding circles of the plane, we may state

THEOREM I. *The necessary and sufficient condition that four points a, b, c, d may form a Hart tetrad is that the harmonically conjugate quadric‡ of the Absolute (qua envelope) with respect to the tetrad may degenerate into a pair of points.*

This interesting result established by Prof. Baker § has been recently supplemented by yet another significant

* Submitted to the twelth conference of the Indian Mathematical Society, held at Aligarh in December 1941.

† I thank Prof. Narasinga Rao for guidance in preparing this paper.

‡ A term due to Baker; *Principles of Geometry*, Vol. III, p. 43.

§ *Principles of Geometry* Vol. IV. pp. 72-75, 82—85.

result due to Dr. Vaidyanathaswamy* relating to complementary Hart tetrads, namely:

THEOREM II. *The eight points of two complementary Hart tetrads lie on the same twisted cubic.*

The purpose of this paper is to point out the connecting link between the two theorems above stated and obtain the geometrical specification of the cubic of Theorem II in relation to a given Hart tetrad and the absolute quadric.

2. Let a, b, c, d be four points in general position in the representative three-space and Ω the absolute quadric. The poles A, B, C, D of the faces bcd, etc. of the tetrahedron $abcd$, say δ, w.r.t Ω determine another tetrahedron $\triangle$ which will be called the *absolute polar tetrahedron* of δ. Now, every member of the net of order quadrics apolar† to $\triangle$ and outpolar to Ω passes through eight points which are the vertices of two tetrahedra $\triangle_1(A_1B_1C_1D_1)$ and $\triangle_2(A_2B_2C_2D_2)$ desmic with $\triangle$. These points are the poles w.r.t Ω of the planes of contact of the eight quadrics circumscribed to δ and having ring contact with Ω. It is well known that these eight points correspond in our representation to the eight circles each of which cuts the four circles a, b, c, d all at the same angle. Thus the four circles that touch a, b, c, d when they belong to a Hart tetrad must accordingly be found among these eight circles.

Now, if four of the above eight circles, say 1, 2, 3, 4, be such that the angle of intersection of every one of them with *any of* the given circles a, b, c, d be the same, then, it follows from what is said above and from the relations between desmic tetrads that the absolute polar tetrad of this tetrad 1, 2, 3, 4 and also of the remaining four form together with δ a desmic system. Reciprocating w.r.t Ω

* *Proc. Ind. Ac. Sci.* 1 (1934), 67-69.

† Apolar means having $\triangle$ for a self-polar tetrahedron.

we find that this amounts to the result that the tetrad of four points in question, namely, 1, 2, 3, 4 together with the remaining tetrad of four points, and the tetrad $\triangle$ form a desmic system. And, as the grouping of the eight points into two tetrads desmic with $\triangle$ is unique, we see that the points 1, 2, 3, 4 must be either the vertices of $\triangle_1$ or $\triangle_2$. If for convenience, we call a tetrad of circles c_1, c_2, c_3, c_4 an *equiangular tetrad* if there exists another tetrad of circles c_1', c_2', c_3', c_4' called the *complementary tetrad* such that every circle of the first tetrad is cut at the same angle by every circle of the second tetrad, we have the following result:

If δ *le an equiangular tetrad, the four points of the complementary tetrad must be the vertices of one or other of the two tetrahedra* $\triangle_1$ *or* $\triangle_2$. (2.1)

Let δ be an equiangular tetrad, $\triangle_1(A_1\, B_1\, C_1\, D_1)$ its complementary tetrad (2.1) and K_a, K_b, K_c, K_d the four quadrics circumscribed to $\triangle_1$ and having ring contact with Ω along its sections with the polar planes X, Y, Z, T w.r.t. Ω of a, b, c, d, the faces of $\triangle$. These four quadrics are linearly independent and are therefore outpolar to six linearly independent class quadrics.

Again, in the pencil of quadrics determined by Q (absolute *qua* locus) and the harmonically conjugate quadric $\overline{Q}$ of Q w.r.t. $\triangle$, there are two special members, namely, a quadric H circumscribed to $\triangle$ and a quadric S apolar to $\triangle$.* Consider the quadrics inpolar to the linear ∞^3 system determined by the quadric H and the quadrics of the net circumscribed to $\triangle_1, \triangle_2$. These may be taken to be three linearly independent quadrics† inscribed to $\triangle$, $\triangle_2$; the quadric Σ w.r.t. which $\triangle, \triangle_1, \triangle_2$ are all self-

* For if $Q \equiv ax^2 + \ldots + 2fyz + \ldots + 2uxt + \ldots$, then $\overline{Q} \equiv ax^2 + \ldots - 2fyz - \ldots - 2uxt - \ldots$ so that $S \equiv Q + \overline{Q}$ and $H \equiv Q - \overline{Q}$.

†That the quadrics inscribed to $\triangle$, $\triangle_2$ are inpolar to H is seen thus: since δ is an equiangular tetrad the quadrics circumscribed to δ, δ_2 are outpolar to Ω. Hence the quadrics inscribed to $\triangle, \triangle_2$ are inpolar to Q, and being also inpolar to S are inpolar to H.

polar and any two quadrics Σ_1, Σ_2 inscribed to $\triangle$, $\triangle_1$ and inpolar to H.

Now, as the quadrics K_a, K_b, K_c, K_d are circumscribed to $\triangle_1$, they are outpolar to the quadrics inscribed to $\triangle, \triangle_2$ and to the quadric Σ, since these quadrics all have $\triangle_1$ for a self-polar tetrahedron. Further Σ_1 is inpolar to H and is inscribed to $\triangle$ apolar to S; it is therefore inpolar to Q. Again Σ_1 touches the faces of $\triangle$, i.e. it is inpolar to the squared planes X^2, Y^2, Z^2, T^2. Hence Σ_1 is inpolar to the system determined by Q and X^2, Q and Y^2, etc., and in particular to the quadrics K_a, K_b, K_c, K_d. Thus Σ_1 and similarly Σ_2 are inpolar to the quadrics K_a, K_b, K_c, K_d. We have thus the result:

The linear ∞^3 system determined by K_a, K_b, K_c, K_d is also the system determined by H and the quadrics of the net circumscribed to $\triangle_1$, $\triangle_2$.

Further, since the quadrics K_a, K_b, K_c, K_d all pass through the vertices of $\triangle_1$, any quadric of the linear system determined by them, and in particular H will pass through the vertices of $\triangle_1$. Now, it is known* that a tetrahedron $\triangle$ inscribed in a quadric H is desmic only to one other tetrahedron inscribed in H; also it is desmic only to one other tetrahedron self-polar to H and the three tetrahedra form a desmic system. Thus, as $\triangle$, $\triangle_1$, $\triangle_2$ belong to a desmic system, we have the following

THEOREM III. *If δ be an equiangular tetrad, the four points of the complementary tetrad are the vertices of the second tetrahedron $\triangle_1$ inscribed in H and desmic to $\triangle$.*

3. We now proceed to apply the results of the previous paragraph to the case when δ is a Hart tetrad. Now, if $\triangle_1$ is the complementary Hart tetrad, then by Theorem I the harmonically conjugate quadric $\overline{Q}$ of Q (absolute *qua* locus), w.r.t. $\triangle$ is a pair of planes, say π, π', intersecting in the line l. Let us denote by Γ

* *Vide* Cubic transformations associated with a desmic system, by Dr. Vaidyanathaswamy, Suppl. to *J. I. M. S.*, (1) 18 (1927), 50.

the desmic point-transformation whose singular points are the vertices of $\triangle$ and fixed points the eight vertices of $\triangle_1$, $\triangle_2$ or symbolically Γ: $[\triangle\ (\triangle_1, \triangle_2)]$. Similarly Γ_1: $[\triangle_1(\triangle, \triangle_2)]$ and Γ_2: $[\triangle_2(\triangle, \triangle_1)]$. We prove the following main theorem, namely

THEOREM IV. *The transform of l in the cubic transformation Γ_1 is the twisted cubic L through the vertices of δ and $\triangle_1$, i.e. through the eight points of the two complementary Hart tetrads.*

Let $\Gamma(a)$ be the transform of a in Γ; then a, $\Gamma(a)$ are conjugate points w.r.t. all the quadrics of the net circumscribed to $\triangle_1$, $\triangle_2$. If δ be a Hart tetrad, all the four quadrics K_a, K_b, K_c, K_d are cones with vertices at a, b, c, d respectively. Now, as the polar plane of a w.r.t. K_a is indeterminate it is obvious that if the polar planes of a w.r.t. K_b, K_c, K_d meet in a', then a, a' are conjugate points for all the cones and therefore to the linear system determined by them. Hence by (2.2), a, a' are conjugate points w.r.t. all the quadrics of the net circumscribed to $\triangle_1$, $\triangle_2$ and also w.r.t. H. Thus a' is $\Gamma(a)$ and further a, $\Gamma(a)$ are also conjugate points w.r.t. H. Also as a, $\Gamma_2(a)$ are always conjugate points w.r.t. H and further $\Gamma(a)$, $\Gamma_1(a)$, $\Gamma_2(a)$* are collinear it follows that a, $\Gamma_1(a)$ are conjugate points w.r.t. H. By parity of reasoning for b, c, d we have the result:

Each vertex of a Hart tetrad δ is conjugate w.r.t H to its transform in the cubic transformation Γ_1. (3.1)

A λ-generator g on H is transformed by Γ into a twisted cubic G through $ABCD$ lying on H; and G cuts the λ-generators in two points. If g passes through A, then G splits up into the μ-generator through A and the conic of intersection of H and the plane BCD; further this conic and the generators through A constitute a fundamental quartic† determined as the intersection of H with

* R. Vaidyanathaswamy *loc. cit.* p. 11.

† *Ibid.*, p. 33.

a quadric through the vertices of $\triangle$, $\triangle_2$. Thus the tangent plane T_A at A to H and the plane BCD together constitute a degenerate quadric in the pencil determined by H and a quadric through $\triangle$, $\triangle_2$. Denoting by T_B, T_C, T_D the tangent planes to H at B, C, D we infer that

The plane pairs X, T_A; Y, T_B; Z, T_C; T, T_D all belong to the linear ∞^3 system determined by H and the quadrics of the net circumscribed to $\triangle$, $\triangle_2$. (3.2)

Consider the polar planes of B w. r. t. the quadrics H, S; Q, $\bar{Q}$, namely, the planes T_B, Y, y and $\bar{Y}$ (say). All these intersect in a line β, say. Further, since the parameters of H and S are harmonic w. r. t. those of Q, $\bar{Q}$ in the pencil determined by them, the pair of planes T_B, Y seperate harmonically the pair y, $\bar{Y}$. Also the polar plane of a w. r. t. the plane pair T_B, Y passes through β and is the harmonic conjugate of the plane (a, β), namely, y w. r. t. the pair T_B, Y; from what is said above it is the plane $\bar{Y}$. Thus the polar planes of a w. r. t. the plane pairs T_B, Y; T_C, Z; T_D, T are respectively the polar planes of B, C, D w. r. t. $\bar{Q}$; and as $\bar{Q}$ is a pair of planes through the line l, these polar planes all pass through l.

Again, the polar planes of A w. r. t. H, S, Q, $\bar{Q}$, namely, T_A, X, x, $\bar{X}$ all pass through a line α; and the polar plane of A w. r. t. $\bar{Q}$ is a plane through l. Hence l meets α in a point a_1, say. Also, the polar plane of a w. r. t. the plane pair T_A, X contains the line α and therefore passes through a_1. Thus a and a_1 are conjugate points w. r. t. the four plane pairs T_A, X; etc. Hence by (3.2), $\Gamma_1(a)$ is the point a_1 and it lies on l. Similarly $\Gamma_1(b)$, $\Gamma_1(c)$, $\Gamma_1(d)$ also lie on l. Hence, the transform of l in Γ_1 is a twisted cubic L through a, b, c, d and through the singular points of Γ_1, namely, the points A_1, B_1, C_1, D_1. We have thus proved that the eight points of two complementary Hart tetrads lie on the same twisted cubic which is the transform of l in Γ_1.

MULTIPLICATIVE FUNCTIONS WHICH ARE FUNCTIONS OF THE *g.c.d.* AND *l.c.m.* OF THE ARGUMENTS*

BY

P. KESAVA MENON, *Annamalainagar.*

[Received 20 March, 1942]

The object of this paper is to give generalizations of two of Dr. R. Vaidyanathaswamy's theorems† on multiplicative functions which are functions of the greatest common divisor (*g.c.d.*) and the least common multiple (*l.c.m*) of the arguments.

All functions considered in this paper are multiplicative functions, i.e. functions which satisfy the condition

$$f(M_1N_1, M_2N_2, \ldots, M_rN_r) = f(M_1, M_2, \ldots, M_r) \times f(N_1, \ldots, N_r), \qquad (1)$$

where ΠM_i is prime to ΠN_i. By the above condition we may write

$$f(M_1, M_2, \ldots, M_r) = \prod_p f(p^{m_1}, p^{m_2}, \ldots, p^{m_r}),$$

where $\prod_p p^{m_i}$ is the resolution into its prime factors of M_i, $(i = 1, 2, \ldots, r)$. The series

$$f_{(p)}(x_1, x_2, \ldots, x_r) = \sum_{\substack{m_i=0 \\ (i=1,2,\ldots,r)}}^{\infty} f(p^{m_1}, p^{m_2}, \ldots, p^{m_r}) x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r}$$

shall be called the generating series to the base p of f. Given the generating series to all prime bases, it follows that the multiplicative function is completely determined.

*This formed part of the thesis for the degree of M.Sc. of the University of Madras and I thank Dr. Vaidyanathaswamy for his guidance.

†The theory of multiplicative arithmetic functions: *Trans. Amer. Math. Soc.* 33 (1931).

The function

$$\sum f_1(d_1, d_2, \ldots, d_r)\, f_2\left(\frac{M_1}{d_1}, \frac{M_2}{d_2}, \ldots, \frac{M_r}{d_r}\right)$$

summed for all divisors d_i of $M_i (i = 1, 2, \ldots, r)$ is called the *composite* of the two functions f_1 and f_2 and is denoted by $(f_1 \cdot f_2)(M_1, \ldots, M_r)$. It can easily be seen that $(f_1 \cdot f_2)$ is also multiplicative, and its generating series to the base p is given by the relation

$$(f_1 \cdot f_2)_{(p)}(x_1, x_2, \ldots, x_r) = f_{1(p)}(x_1, x_2, \ldots, x_r)\, f_{2(p)}(x_1, x_2, \ldots, x_r).$$

The following special functions are required in the sequel.

A *linear function* is one for which (1) is satisfied for all values of the arguments M_i and N_i. It follows from this that the generating series of a linear function to the prime base p is the expansion of

$$1 / \prod_{i=1}^{r} (1 - a_{pi}\, x_i),$$

where $a_{pi} = f(1, 1, \ldots, p, \ldots, 1)$, p in the ith place and 1 elsewhere.

A *principal function* is defined as the function $f(M_1, M_2, \ldots, M_r)$ which vanishes whenever two of its arguments are unequal. The function $\psi(M) = f(M, M, \ldots, M)$ is called the function of one argument equivalent to the principal function f, and the relation between them is indicated by writing $f = \text{princ}\ \psi$. Similarly the function $f(M_1, M_2, \ldots, M_r)$ is said to be the *semiprincipal function* equivalent to $\psi(M)$, if, for every prime p, $f(p^{t_1}, p^{t_2}, \ldots, p^{t_r})$ vanishes whenever any two non-zero exponents t are unequal and is equal to $(-1)^{r-s+1}\psi(p^u)$ if s of the exponents t are zero and the remaining are equal to u. In this case f is denoted by semiprinc ψ.

It is readily seen that if $\psi_{(p)}(x)$ is the generating function to the base p of $\psi(M)$, then the generating functions of the principal and semiprincipal functions of r arguments equivalent to $\psi(M)$ are given by

$$(\text{princ}\ \psi)_{(p)}(x_1, x_2, \ldots, x_r) = \psi_{(p)}(x_1 x_2 \ldots x_r),$$

and

$$(\text{semiprinc } \psi)_{(p)}(x_1, \ldots, x_r) = \sum_i \psi_{(p)}(x_i) - \sum_{i,j} \psi_p(x_i x_j) + \ldots$$
$$+ (-1)^r \psi_{(p)}(x_1 x_2 \ldots x_r).$$

The above definitions and symbolism are due to Vaidyanathaswamy. We shall further define the *convolute* with respect to $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ of a function f by the relations

$$\underset{(\lambda_1,\ldots,\lambda_r)}{\text{conv}} f(M_1, M_2, \ldots, M_r) = f(M_1^{1/\lambda_1}, M_2^{1/\lambda_2}, \ldots, M_r^{1/\lambda_r}),$$
$$\text{if } M_i \text{ is a } \lambda_i\text{th power } (i = 1, 2, \ldots, r)$$
$$= 0, \text{ otherwise.}$$

Evidently if $f_{(p)}(x_1, x_2, \ldots, x_r)$ is the generating function of f, then the generating function of $\underset{(\lambda_1,\ldots,\lambda_r)}{\text{conv}} f$ is $f_{(p)}(x_1^{\lambda_1}, x_2^{\lambda_2}, \ldots, x_r^{\lambda_r})$.

We proceed to prove

THEOREM 1. *The necessary and sufficient condition that $f(M_1, M_2, \ldots, M_r)$ is a function $\psi(g)$ of the g.c.d. g of $M_1^{(\lambda_1)}$, $M_2^{(\lambda_2)}, \ldots, M_r^{(\lambda_r)}$, where $M_i^{(\lambda_i)}$ is the greatest divisor of M_i which is a λ_ith power, is that $f = E \,.\, \underset{(\lambda_1,\ldots\lambda_r)}{\text{conv}}\ \text{princ}\ (\psi \,.\, E^{-1})$, $E(M_1, \ldots, M_r)$ being the function which takes the value 1 for all values of the arguments and E^{-1} the function which when composed with E gives a function which vanishes except when all the arguments are 1.*

PROOF. Let the generating series to the base p of f and ψ be

$$f_{(p)}(x_1, x_2, \ldots, x_r) = \Sigma a(m_1, m_2, \ldots, m_r)\, x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r}$$

and
$$\psi_{(p)}(x) = \sum_{m=0}^{\infty} A(m) x^m.$$

From the hypothesis on the function f it follows that $a(m_1, m_2, \ldots, m_r) = A(m)$, if m is the least of $[m_1/\lambda_1]$, $[m_2/\lambda_2]$, $\ldots$, $[m_r/\lambda_r]$, where $[n]$ denotes the integral part of n. Hence by a rearrangement of terms we find that

$$f_p(x_1,\ldots,x_r) = \sum_{m=0}^{\infty} A(m)\Big[\sum_{i_1,\ldots,i_r=0}^{\infty} x_1^{(m\lambda_1+i_1)} x_2^{(m\lambda_2+i_2)}\ldots x_r^{(m\lambda_r+i_r)}$$

$$- \sum_{i_1,\ldots,i_r=0}^{\infty} x_1^{(\overline{m+1}\,\lambda_1+i_1)} x_2^{(\overline{m+1}\lambda_2+i_2)}\ldots x_r^{(\overline{m+1}\lambda_r+i_r)}\Big]$$

$$= \sum_{m=0}^{\infty} A(m)\ (x_1^{\lambda_1} x_2^{\lambda_2}\ldots x_r^{\lambda_r})^m (1-x_1^{\lambda_1}\ldots x_r^{\lambda_r})\Big[\sum_{i_1,\ldots,i_r=0}^{\infty} x_1^{i_1}x_2^{i_2}\ldots x_r^{i_r}\Big]$$

$$= \frac{(1-x_1^{\lambda_1}\ldots x_r^{\lambda_r})}{\prod_{i=1}^{r}(1-x_i)} \sum_{m=0}^{\infty} A(m)\ (x_1^{\lambda_1} x_2^{\lambda_2}\ldots x_r^{\lambda_r})^m,$$

which is the generating series of $E\ .\ \underset{(\lambda_1,\ldots,\lambda_r)}{\text{conv}}\ \text{princ}\ (\psi\ .\ E^{-1})$.

COR. 1. Taking $\lambda_1=\lambda_2=\ldots=\lambda_r=1$, we get Vaidyanathaswamy's theorem:

The necessary and sufficient condition that $f(M_1,\ldots,M_r)$ should be a function $\psi(g)$ of the g.c.d. g of $M_1, M_2,\ldots, M_r$ is that $f=E$. princ $(\psi\ .\ E^{-1})$.

COR. 2. *Let $\phi_k(N)$ be Jordan's function representing the number of sets of k numbers not greater than N whose g.c.d. is prime to N, $\tau(N)$ the number of divisors of N, and $\sigma_k(N)$ the sum of the kth powers of the divisors of N. Then*

$$\sum\phi_{k_1}\Big(\frac{M_1}{\delta_1}\Big)\ldots\phi_{k_r}\Big(\frac{M_r}{\delta_r}\Big)\tau(\delta) = M_1^{k_1}\ldots M_r^{k_r}\frac{\sigma_{k_1+k_2+\ldots+k_r}(g)}{g^{k_1+k_2+\ldots+k_r}},$$

where the summation is extended over all divisors δ_i of M_i, δ being the the g.c.d. of $\delta_1^{(\lambda_1)},\ldots,\delta_r^{(\lambda_r)}$ and g being the g.c.d. of $M_1^{(\lambda_1)}, M_2^{(\lambda_2)},\ldots, M_r^{(\lambda_r)}$.

For, writing $I_{k_1,k_2,\ldots,k_r}\ (M_1,\ldots,M_r) = M_1^{k_1}\ldots M_r^{k_r}$ and $\Phi_{k_1,\ldots,k_r}\ (M_1,\ldots,M_r) = (I_{k_1,\ldots,k_r}\ .\ E^{-1})\ (M_1,\ldots,M_r)$, $= \prod_{i=1}^{r}\phi_{k_i}\ (M_i)$, we have

$$\Phi_{k_1,\ldots,k_r}\ .\ E^2(g)^*$$

$$= \{\Phi_{k_1,\ldots,k_r}\ .\ E\ .\ \underset{\lambda_1,\ldots,\lambda_r}{\text{conv}}\ \text{princ}\ (E^2\ .\ E^{-1}\}\ (M_1,\ldots,M_r)$$

*E^2 denotes $E\ .\ E$, the composite of E with itself.

$$= \left\{ I_{k_1,\ldots,k_r} \cdot \underset{(\lambda_1,\ldots,\lambda_r)}{\text{conv}} \text{ princ } E \right\} (M_1,\ldots, M_r)$$

$$= M_1^{k_1} \ldots M_r^{k_r} \Sigma d^{-(k_1+\ldots+k_r)}, \text{ summed for divisors } d \text{ of } g,$$

$$= M_1^{k_1} \ldots M_r^{k_r} \frac{\sigma_{k_1+\ldots+k_r}(g)}{g^{k_1+\ldots+k_r}}.$$

THEOREM 2. *The necessary and sufficient condition that* $f(M_1,\ldots, M_r)$ *be a function* $\psi(l)$ *of the l.c.m. of* $M_1^{(\lambda_1)},\ldots, M_r^{(\lambda_r)}$, *where* $M_i^{(\lambda_i)}$ *is defined as in Theorem* I, *is that*

$$f = E . \text{conv semiprinc } (\psi . E^{-1}).$$

For, let the generating series to the base p of f and ψ be the same as in Theorem I. We have evidently $a(m_1,\ldots, m_r) = A(m)$, where m is the largest of $[m_1/\lambda_1]$, $[m_2/\lambda_2], \ldots, [m_r/\lambda_r]$. Hence by a rearrangement of terms,

$$f_{(p)}(x_1,\ldots, x_r) = \sum_{m=0}^{\infty} A(m) \Bigg[\sum_{i_1=0}^{\lambda_1(m+1)-1} \cdots \sum_{i_r=0}^{\lambda_r(m+1)-1} x_1^{i_1} \ldots x_r^{i_r} - \sum_{i_1=0}^{\lambda_1 m-1} \cdots \sum_{i_r=0}^{\lambda_r m-1} x_1^{i_1} \ldots x_r^{i_r} \Bigg],$$

$$= \sum_{m=0}^{\infty} A(m) \left[\prod_{i=1}^{r} \frac{(1-x_i^{\lambda_i(m+1)})}{(1-x_i)} - \Pi \frac{(1-x_i^{\lambda_i m})}{(1-x_i)} \right]$$

$$= \frac{1}{\prod_{i=1}^{r} (1-x_i)} \sum A(m) \Big[\sum_i x_i^{\lambda_i m} (1-x_i^{\lambda_i}) - \sum_{i,j} (x_i^{\lambda_i} x_j^{\lambda_j})^m (1-x_i^{\lambda_i} x_j^{\lambda_j}) + \ldots \Big]$$

$$= \frac{1}{\prod_{i=1}^{r} (1-x_i)} \Big[\sum_i (\psi . E^{-1})_{(p)} (x_i^{\lambda_i}) - \sum_{i,j} (\psi . E^{-1})_{(p)} (x_i^{\lambda_i} x_j^{\lambda_j}) + \ldots \Big],$$

from which the required result follows.

COR. I. *The case when* $\lambda_1 = \ldots = \lambda_r = 1$ *is Vaidyanathaswamy's theorem.*

Let $L(M)$ be the linear function whose generating series to the base p is $L_{(p)}(x) = 1/(1-ax)$. Also let

$\phi_p(x) = (1-x)/(1-ax)$ be the generating function of $\phi(M)$ so that

$$(\text{semiprinc } \phi)_{(p)}(x, y) = \frac{1-x}{1-ax} + \frac{1-y}{1-ay} - \frac{1-xy}{1-axy}$$
$$= \frac{(1-a^2xy)(1-x)(1-y)}{(1-ax)(1-ay)(1-axy)},$$

or semiprinc $\phi = \phi\phi$. princ $[L.\ (L \times L)^{-1}]$,

where $\phi\phi(M, N)$ denotes the function $\phi(M)\,\phi(N)$. Now

$$L(l) = \{ E . \underset{(\lambda,\,\mu)}{\text{conv}} \text{ semiprinc } (L . E^{-1} \}(M, N),$$

where l is the *l.c.m.* of $M^{(\lambda)}, N^{(\mu)}$. Therefore

$$L(l) = E . \underset{(\lambda,\,\mu)}{\text{conv}} [\phi\phi \text{ . princ } (L . (L \times L)^{-1})],$$

or, $L(l) . \underset{(\lambda,\,\mu)}{\text{conv}} [E \text{ . princ } (L \times L)] = E . \underset{(\lambda,\,\mu)}{\text{conv}} (LL \text{ . princ } L)$,

where $LL(M, N)$ denotes $L(M)\,L(N)$; so that

$$\{ L(l) \}_{(p)} (x, y) = \frac{(1-x^\lambda)(1-y^\mu)(1-a^2x^\lambda y^\mu)}{(1-x)\ (1-y)\ (1-ax^\lambda)\ (1-ay^\mu)\ (1-ax^\lambda y^\mu)}.$$

Taking $L(M, N) = I_a(M, N) = (MN)^a$, we have

COR. 2.

$$I_a(l) . \underset{(\lambda,\,\mu)}{\text{conv}} [E \text{ . princ } I_{2a}] = E . \underset{(\lambda,\,\mu)}{\text{conv}} [I_a \text{ . princ } I_a]$$

or,

$$\sum \left\{ l\left(\frac{M}{\delta_1}, \frac{N}{\delta_2}\right) \right\}^a \underset{(\lambda,\,\mu)}{\text{conv}}\ \sigma_{2a} \left\{ g(\delta_1, \delta_2) \right\}$$
$$= \sum \sigma_a \left\{ g(\delta_1^{(\lambda)}, \delta_2^{(\mu)}) \right\} . \underset{(\lambda,\,\mu)}{\text{conv}} \left(\frac{MN}{\delta_1\, \delta_2} \right)^a,$$

where the summations are for all divisors δ_1, δ_2 *of* M, N *respectively.*

Theorems 1 and 2 may be extended to functions of a matrix set of arguments which give generalizations of corresponding theorems of Vaidyanathaswamy.

TRANSFORMATIONS OF ARITHMETIC FUNCTIONS*

BY

P. KESAVA MENON, *Annamalai University.*

[Received 24 May, 1942].

1. An arithmetic function $f(M_1, M_2, \ldots, M_r)$ of r arguments is one which is defined for all non-zero positive integral values of the arguments $M_1, M_2, \ldots, M_r$. f is called multiplicative if

$$f(M_1N_1, M_2N_2, \ldots, M_rN_r) = f(M_1, M_2, \ldots, M_r) \times f(N_1, N_2, \ldots, N_r) \qquad (1)$$

when ΠM_i is prime to ΠN_i. f is called linear if (1) holds for all values of M_i and N_i.

Let $f(M)$ be an arithmetic function of a single argument, and let $L_1(M)$, $L_2(M)$ be two linear functions, $L_2(M) \neq 0$ for any M. Suppose $M = p_1^{m_1} \ldots p_t^{m_t}$ and $N = p_1^{n_1} \ldots p_t^{n_t}$ are the factorizations of M and N into powers of different primes, and let $T(M, N)$ denote the product of the binomial coefficients $\Pi \binom{m_i - 1}{n_i - 1}$. Then $T(M, N)$ is a multiplicative function which vanishes if and only if one of the factors vanishes, i.e. $T(M, N) \neq 0$ unless N is a factor of M and is divisible by all the prime factors of M. Moreover $T(M, M) = 1$.

Let us also define the linear function $\lambda_k(M) = k^\nu$, where ν is the number of prime factors of M.

We shall now study a transformation of $f(M)$. Let

$$\sum_{\delta/M} L_1\left(\frac{M}{\delta}\right) L_2(\delta)\; T(M, \delta)\; f(\delta) \qquad (2)$$

*This is a modified form of part of the thesis for the M.Sc. degree of the Madras University, and I thank Dr. Vaidyanathaswamy for his guidance,

be denoted by $f_1(M)$. We shall call f_1 the *transform* of f. It is obvious that we may write (2) in the form

$$\sum_{\delta=1}^{M} L_1\left(\frac{M}{\delta}\right) L_2(\delta)\, T(M, \delta)\, f(\delta). \tag{3}$$

Writing $f(\delta) = x_\delta$, $f_1(\delta) = y_\delta$, $L_1(M/\delta)\, L_2(\delta)\, T(M, \delta) = a_{M,\delta}$, we have from (3) an infinite system of linear relations

$$\sum_{j=1}^{i} a_{i,j}\, x_j = y_i \ (i = 1, 2, 3, \ldots), \tag{4}$$

$a_{i,j} = 0$ for $j > i$ and $a_{i,i} = L_2(i) \neq 0$.

Hence we may express the x's as linear functions of the y's by the relations

$$\sum_{j=1}^{i} b_{i,j}\, y_j = x_i \ (i = 1, 2, 3, \ldots). \tag{5}$$

The coefficients in the first n of the relations (5) depend only on those of the first n of the relations (4) for every n. That is, from the n equations

$$\sum_{j=1}^{i} a_{i,j}\, x_j = y_i \ (i = 1, 2, \ldots, n)$$

we get

$$\sum_{j=1}^{i} b_{i,j}\, y_j = x_i \ (i = 1, 2, \ldots, n)$$

for all n. Suppose we determine a set of coefficients $c_{i,j}$ such that, for each n, the equations

$$\left.\begin{aligned} \sum_{j=1}^{i} a_{i,j} x_j &= y_i \\ \sum_{j=1}^{i} c_{i,j}\, y_j &= x_i \end{aligned}\right\} \quad (i = 1, 2, \ldots, n)$$

are satisfied by an infinity of distinct systems of x's and y's. Then we may assert that $c_{i,j} = b_{i,j}$.

By the foregoing, it follows that to find the form of the expression for $f_1(M)$ in terms of $f(M)$ we may assume $f(M)$ to be multiplicative, since multiplicative functions form an infinite class. Let, therefore

$$f(M) = \Pi f(p^m).$$

Then it can be shown that $f_1(M)$ is also multiplicative, so that

$$f_1(M) = \Pi f_1(p^m).$$

$$\text{But } f_1(p^m) = \sum_{\delta/p^m} L_1\left(\frac{p^m}{\delta}\right) L_2(\delta)\ T(p^m, \delta) f(\delta)$$

$$= \sum_{i=0}^{m} L_1(p^{m-i})\ L_2(p^i) \binom{m-1}{i-1} f(p^i)$$

$$= \sum_{i=0}^{m} \binom{m-1}{i-1} \alpha^i c^{m-i} f(p^i), \tag{6}$$

where c, α are written for $L_1(p)$, $L_2(p)$ respectively.

If the series

$$1 + f(p)\ x + f(p^2)\ x^2 + \ldots$$

called the generating series to the base p of $f(M)$, be denoted by $f_{(p)}(x)$, then the generating series $f_{1(p)}(x)$ to the base p of $f_1(M)$ will be given by

$$f_{1(p)}(x) = 1 + \sum_{m=1}^{\infty} \left[\sum_{i=0}^{m} \binom{m-1}{i-1} \alpha^i c^{m-i} f(p^i)\right] x^m$$

$$= f_{(p)}\left(\frac{\alpha x}{1 - cx}\right). \tag{7}$$

Writing $x/(\alpha + cx)$ for x in (7) we get

$$f_{(p)}(x) = f_{1(p)}\left(\frac{x}{\alpha + cx}\right)$$

$$= 1 + \sum_{m=1}^{\infty} f_1(p^m) \left(\frac{x}{\alpha + cx}\right)^m$$

$$= 1 + \sum_{m=1}^{\infty} \left[\sum_{i=0}^{m} \binom{m-1}{i-1} \frac{(-c)^{m-i}}{\alpha^m} f_1(p^i)\right] x^m,$$

so that

$$f(p^m) = \sum_{i=0}^{m} \binom{m-1}{i-1} \frac{(-c)^{m-i}}{\alpha^m} f_1(p^i).$$

Therefore

$$f(M) = \prod_p \sum_{i=0}^{m} \binom{m-1}{i-1} (-1)^{m-i} c^{m-i} f_1(p^i)/\alpha^m$$

$$= \frac{1}{L_2(M)} \sum_{\delta/M} L_1\left(\frac{M}{\delta}\right) \lambda_{-1}\left(\frac{M}{\delta}\right) T(M, \delta) f_1(\delta)$$

$$= \frac{1}{L_2(M)} \sum_{\delta=1}^{M} L_1\left(\frac{M}{\delta}\right) \lambda_{-1}\left(\frac{M}{\delta}\right) T(M, \delta) f(\delta).$$

It follows that

$$b_{M, \delta} = \frac{1}{L_2(M)} \times L_1\left(\frac{M}{\delta}\right) \lambda_{-1}\left(\frac{M}{\delta}\right) T(M, \delta).$$

Thus we have

THEOREM 1. *If*

$$f_1(M) = \sum_{\delta/M} L_1\left(\frac{M}{\delta}\right) L_2(\delta) T(M, \delta) f(\delta),$$

then

$$f(M) = \frac{1}{L_2(M)} \sum_{\delta/M} L_1\left(\frac{M}{\delta}\right) \lambda_{-1}\left(\frac{M}{\delta}\right) T(M, \delta) f_1(\delta).$$

COR. 1. *Taking $L_1(M) = L_2(M) = E(M)$, the function whose value is unity for all values of M, if*

$$f_1(M) = \sum_{\delta/M} T(M, \delta) f(\delta),$$

then

$$f(M) = \sum_{\delta/M} \lambda_{-1}\left(\frac{M}{\delta}\right) T(M, \delta) f_1(\delta).$$

COR. 2. *Taking $L_2(M) = \lambda_{-1}(M)$ in Theorem 1, if*

$$f_1(M) = \sum_{\delta/M} L_1\left(\frac{M}{\delta}\right) \lambda_{-1}(\delta) T(M, \delta) f(\delta), \qquad (8)$$

then

$$f(M) = \sum_{\delta/M} L_1\left(\frac{M}{\delta}\right) \lambda_{-1}(\delta)\ T(M, \delta)\ f_1(\delta).$$

So we see that f and f_1 may be interchanged in (8), and for this reason we call f and f_1 reciprocal functions with respect to the linear function L_1.

2. Let us take $L_2(M) = \lambda_k(M)$ in (2) and define the sequence of functions $f_0(=f), f_1, f_2, \ldots$ as follows:—

$$f_i(M) = \sum_{\delta/M} L\left(\frac{M}{\delta}\right) \lambda_k(\delta)\ T(M, \delta)\ f_{i-1}(\delta)\ (i = 1, 2, 3, \ldots),$$

L being an arbitrary linear function. It is clear that $f_i(M)$ can be expressed linearly in terms of $f(d)$, $(d = 1, 2, \ldots, M)$.

To determine the coefficients we may therefore assume, as in Theorem 1, $f(M)$ to be multiplicative. If $f_{i(p)}(x)$ denote the generating series to the base p of $f_i(M)$, $f_{i(p)}(x)$ is obtained from $f_{i-1(p)}(x)$ in exactly the same way as $f_{1(p)}(x)$ is obtained from $f_{(p)}(x)$. Thus we get

$$f_{i(p)}(x) = f_{i-1(p)}\left(\frac{kx}{1-cx}\right)\ (i = 1, 2, 3, \ldots),$$

from which it follows that

$$f_{i(p)}(x) = f_{(p)}\left\{\frac{k^i x}{1-c(1+k+\ldots+k^{i-1})x}\right\}.$$

This gives

$$f_i(p^m) = \sum_{j=0}^{m} \binom{m-1}{j-1} k^{ij} c^{m-j} [i]^{m-j} f(p^j),$$

where $[i]$ denotes $(k^i - 1)/(k-1)$, and so

$$f_i(M) = \sum_{\delta/M} L\left(\frac{M}{\delta}\right) \lambda_{[i]}\left(\frac{M}{\delta}\right) \{\lambda_k(\delta)\}^i\ T(M, \delta)\ f(\delta).$$

This result may be stated as

THEOREM 2. *If $f_0(=f), f_1, f_2, \ldots$ be a sequence of functions defined by*

$$f_i(M) = \sum_{\delta/M} L\left(\frac{M}{\delta}\right) \lambda_k(\delta)\ T(M, \delta) f_{i-1}(\delta), \ (i = 1, 2, 3, \ldots)$$

then

$$f_i(M) = \sum_{\delta/M} L\left(\frac{M}{\delta}\right) \lambda_{[i]}\left(\frac{M}{\delta}\right) \{\lambda_k(\delta)\}^i\ T(M, \delta) f(\delta).$$

COR. 1. *Taking $k = 1$, $L(M) = E(M)$; if $f_0(=f)$, $f_1, f_2, \ldots$ be defined by*

$$f_i(M) = \sum_{\delta/M} T(M, \delta)\ f_{i-1}(\delta), \ (i = 1, 2, 3, \ldots)$$

then

$$f_i(M) = \sum_{\delta/M} \lambda_i\left(\frac{M}{\delta}\right)\ T(M, \delta) f(\delta).$$

COR. 2. *Taking k to be an nth root of unity other than 1, and noting that $\lambda_{[n]}(M)$ vanishes except when $M = 1$, and that $\{\lambda_k(M)\}^n = 1$, we have $f_n(M) = f(M)$.*

The functions $f_0, \ldots, f_{n-1}$ may be called a cyclic system of order n with respect to the linear function L. Reciprocal functions form a cyclic system of order 2.

3. Let $F(M)$ denote the composite of $f(M)$ and $\phi(M)$, i.e.

$$F(M) = \sum_{\delta/M} f(\delta)\, \phi\left(\frac{M}{\delta}\right),$$

and let F_1, f_1, ϕ_1 be the transforms by (2) of F, f, ϕ respectively. It is natural to ask how the composite of f_1 and ϕ_1 is related to F_1. Here again, as in Theorem 1, we may assume f and ϕ and hence also F to be multiplicative. It is seen that the generating series of F, the composite of f and ϕ, to the base p, is given by

$$F_{(p)}(x) = f_{(p)}(x) \times \phi_{(p)}(x).$$

Hence the generating series to the base p of F_1 is

$$F_{1(p)}(x) = F_{(p)}\left(\frac{\alpha x}{1-cx}\right) = f_{(p)}\left(\frac{\alpha x}{1-cx}\right) \times \phi_{(p)}\left(\frac{\alpha x}{1-cx}\right)$$
$$= f_{1(p)}(x) \times \phi_{1(p)}(x),$$

which is the generating series of the composite of f_1 and ϕ_1 to the base p. It follows that F_1 is the composite of f_1 and ϕ_1. This is stated as

THEOREM 3. *If*

$$F(M) = \sum_{\delta|M} f(\delta)\,\phi\left(\frac{M}{\delta}\right),$$

$$f_1(M) = \sum_{\delta|M} L_1\left(\frac{M}{\delta}\right) L_2(\delta)\; T(M, \delta)\, f(\delta),$$

and $\phi_1(M)$ and $F_1(M)$ are defined by similar relations, then

$$F_1(M) = \sum_{\delta|M} f_1(\delta)\;\phi_1\left(\frac{M}{\delta}\right).$$

4. An analogous result holds for the compound of two functions. The compound $\psi(M)$ of $f(M)$ and $\phi(M)$ is defined by

$$\psi(M) = \Sigma f(d)\,\phi(M/d)$$

where the summation on the right runs over all divisors d of M which are prime to M/d. If f, ϕ be multiplicative then ψ is also multiplicative and its generating series to the base p is given by

$$\psi_{(p)}(x) = f_{(p)}(x) + \phi_{(p)}(x) - 1.$$

By means of this we get

THEOREM 4. *The transform by* (2) *of the compound of two functions is the compound of the transforms by* (2) *of the functions.*

From Theorems 3 and 4 it follows that the reciprocal of the composite (or compound) of a function and its reciprocal is the composite (or compound) itself, the

reciprocals being with respect to the same linear function. Functions having this property may be called *self-reciprocal*.

This result may be generalized and written as

THEOREM 5. *If $f_0, f_1, \ldots, f_{n-1}$ be a cyclic system of functions of order n with respect to the linear function L, and F and ψ denote respectively the continued composite and compound of $f_0, f_1, \ldots, f_{n-1}$, then*

$$F(M) = \sum_{\delta/M} L\left(\frac{M}{\delta}\right) \lambda_k(\delta)\ T(M, \delta)\ F(\delta)$$

and

$$\psi(M) = \sum_{\delta/M} L\left(\frac{M}{\delta}\right) \lambda_k(\delta)\ T(M, \delta)\ \psi(\delta),$$

where k is the nth root of unity occurring in the transformations.

The proof depends on the fact that the generating series to the base p of the continued composite and compound respectively of $f_0, f_1, \ldots, f_{n-1}$ are

$$\prod_{i=0}^{n-1} f_{(p)}\left\{\frac{k^i x}{1-c[i]x}\right\}$$

and

$$\sum_{i=0}^{n-1} f_{(p)}\left\{\frac{k^i x}{1-c[i]x}\right\} - (n-1),$$

where $f_{(p)}(x)$ is the generating series of $f(=f_0)$ and $[i]$ denotes $(k^i-1)/(k-1)$, and that the transformation is equivalent to substituting $kx/(1-cx)$ for x in the above expressions. Such a substitution leaves the expressions unaltered from which we assert the truth of the theorem.

5. Transformations of the type studied above may be found for functions of several arguments. Here again, in establishing the various results we may assume the functions to be multiplicative, and the truth of the results for the non-multiplicative case may be inferred as for Theorem 1.

Let $L_i(M)$, $L'_i(M)$ $(i = 1, 2, \ldots, r)$ be linear functions of a single argument M and $f(M_1, \ldots, M_r)$ a general arithmetic function. Let $T\left(\begin{matrix} M_1, \ldots, M_r \\ N_1, \ldots, N_r \end{matrix}\right)$ denote the multiplicative function

$$\prod_p \frac{(\Sigma m_i - 1)!}{(\Sigma n_i - 1)\ \Pi\,(m_i - n_i)!},$$

where $\prod_p p^{m_i}$, $\prod_p p^{n_i}$ are the resolutions into their prime factors of M_i and N_i $(i = 1, 2, \ldots, r)$ respectively. Then we have

THEOREM 6. *If*

$$f_1(M_1, \ldots, M_r) = \sum_{\substack{\delta_i/M_i \\ (i=1, 2, \ldots, r)}} \left\{ \prod_i L_i\left(\frac{M_i}{\delta_i}\right) L_i'(\delta_i) \right\} \times T\left(\begin{matrix} M_1, \ldots, M_r \\ \delta_1, \ldots, \delta_r \end{matrix}\right) f(\delta_1, \ldots, \delta_r),$$

then $$f(M_1, M_2, \ldots, M_r) = \frac{1}{\prod_i L_i'(M_i)} \sum_{\substack{\delta_i/M_i \\ (i=1, 2, \ldots, r)}} \left\{ \Pi L_i\left(\frac{M_i}{\delta_i}\right) \right\} \times \lambda_{-1}\left(\frac{M_1 M_2 \ldots M_r}{\delta_1 \delta_2 \ldots \delta_r}\right) T\left(\begin{matrix} M_1, \ldots, M_r \\ \delta_1, \ldots, \delta_r \end{matrix}\right) f_1(\delta_1, \ldots, \delta_r).$$

For, if f be multiplicative then f_1 is also multiplicative, and if the series

$$f_{(p)}(x_1, x_2, \ldots, x_r) = \Sigma f(p^{m_1}, \ldots, p^{m_r})\, x_1^{m_1} x_2^{m_2} \ldots x_r^{m_r}$$

be called the generating series to the base p of f, then it can be shown that the generating series to the base p of f_1 is

$$f_{1(p)}(x_1, \ldots, x_r) = f_{(p)}\left\{\frac{a_1 x_1}{1 - \Sigma b_i x_i}, \ldots, \frac{a_r x_r}{1 - \Sigma b_i x_i}\right\},$$

where $a_i = L_i'(p)$ and $b_i = L_i(p)$, so that

$$f_{(p)}(x_1, x_2, \ldots, x_r) = f_{1(p)}\left\{\frac{x_1/a_1}{1 + \Sigma \frac{b_i x_i}{a_i}}, \ldots, \frac{x_r/a_r}{1 + \Sigma \frac{b_i x_i}{a_i}}\right\},$$

from which the theorem follows.

The repetition n times of the transformation in Theorem 6 is seen, in the case of multiplicative functions, to be equivalent to the substitutions

$$a_i^n x_i / \left\{ 1 - \sum_s b_s (1 + a_s + \dots + a_s^{n-1})\, x_s \right\} \text{ for } x_i,\ (i = 1, 2, \dots, r)$$

in the generating series of f to obtain the generating series of f_n. These will reduce to identical substitutions if $a_s = 1$, $b_s = 0$, or $a_s =$ an nth root of unity other than 1, $(s = 1, 2, \dots, r)$.

This helps us to extend the concept of cyclic and reciprocal systems to multiplicative functions of several arguments.

Another transformation of f is given by the following

THEOREM 7. *If*

$$f_1(M_1, \dots, M_r) = \sum_{\substack{\delta_i / M_i \\ (i = 1, 2, \dots, r)}} \left\{ \prod_i \lambda_{-1}(\delta_i)\ \ T(M_i, \delta) \right\} f(\delta_1, \delta_2, \dots, \delta_r)$$

then the relation is a reciprocal one between f and f_1.

For if $f_{(p)}\ (x_1, \dots, x_r)$ be the generating series to the base p of f, then the generating series to the same base of f_1 is given by

$$f_{1(p)}(x_1, x_2, \dots, x_r) = f_{(p)} \left\{ \frac{x_1}{x_1 - 1}, \frac{x_2}{x_2 - 1}, \dots, \frac{x_r}{x_r - 1} \right\}.$$

129116
-600 008
JL
Bm21, N34
N34.6.3

Publications of the Indian Mathematical Society

1. Memoir on Cubic Transformations Associated with a Desmic System, by Dr. R. Vaidyanathaswami, pp. 92, Price Rs. 3.
2. Tables of Partitions, by Dr. Hansraj Gupta, pp. 81. Price Rs. 5.

Back Numbers of the *Journal* and of the *Student* are available, except for a few numbers out of stock.

THE MATHEMATICAL ASSOCIATION OF AMERICA

The Mathematical Association of America is devoted primarily to the interests of collegiate mathematics. It now has over two thousand individual and institutional members. There are twenty-two sections at present, representing thirty-one different States. The Association holds two national meetings per year and the sections hold one or two meetings per year. All meetings, both national and sectional, are reported in the official Journal

Application blanks for membership may be obtained from the Secretary, Prof. W.D. Cairns, Oberlin, Ohio.

THE AMERICAN MATHEMATICAL MONTHLY

Official Organ of

THE MATHEMATICAL ASSOCIATION OF AMERICA

Is the Only Journal of Collegiate Grade in the Mathematical Field in this Country.

Most of its mathematical contributions can be read and understood by those who have not specialized in mathematics beyond the Calculus.

The historical papers, which are numerous and of high grade, are based upon original research.

The QUESTIONS and DISCUSSIONS, which are timely and interesting, cover a wide variety of topics.

Surveys of the contents of recent books constitute a valuable guide to current mathematical literature.

The topics in the department of Undergraduate Mathematics Clubs have excited wide interest among those engaged in teaching mathematics.

The NEWS and NOTICES cover a wide range of interest and information.

The Problems and Solutions hold the attention and activity of a large number of persons who are lovers of Mathematics for its own sake.

PRINTED BY R. NARAYANASWAMI IYER AT THE MADRAS LAW JOURNAL PRESS, MUNDAKANNI AMMAN KOIL STREET, MYLAPORE, MADRAS.